Marian Muresan

A Concrete Approach to Classical Analysis

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**Preface**

This book reflects the conclusions of the author to some simple questions: “What should an easy comprehensible introduction to classical mathematical analysis look like? Can we avoid the basic results on differential and integral calculus to jump into abstract results? Actually, which results are considered as basic? Is the book a bridge to some new topics of research?” The influence of functional analysis and Bourbakism has been clear for a long time. At the same time, numerical methods emerged from analysis. It is hard to imagine discrete mathematics without analysis. New and even unexpected tendencies appeared. It is enough to mention some of them, experimental mathematics and scientific computing. These two topics are illustrated by two remarkable books [25] and [20].

Our answer to all these questions consists of our somehow taking all these fields into account. We mean, on the solid ground of classical results (sets, functions, metric spaces, sequences, series, limits, continuity, differentiability, and integrability) that we have to introduce newer results. Why introduce new results? They forcefully appear every day. Moreover, new and incredi- ble methods appear. We mention only two of them presently considered as belonging to “experimental mathematics,” namely the fast computation of the *π* number based on BBP methods, Ramanujan methods. Other methods explore strange functions by computers, that is, the nowhere differentiable functions. The latter topic has been considered as one belonging to “pure mathematics.” Presently it came down into the laboratory of mathematical experiments. This means that by experimental methods we catch a result and then prove it rigorously.

We are pressed to take into account some parts from mathematics and to neglect many others. The present book is focused on differential and integral calculus.

Mathematical analysis offers a solid ground to many achievements in ap- plied and discrete mathematics. In spite of the fact that this book concerns part of what is customarily called mathematical analysis, we have tried to include useful and relevant examples, exercises, and results enlightening the

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reader on the power of mathematical analysis tools. In this respect the topics covered by our book are quite “concrete.”

The strong interplay between so-called theoretical mathematics and scien- tific computing has been emphasized by D. H. Bailey as “To this day I live in two worlds, theoretical math and scientific computing. I’m trying to marry these two by applying advanced computing to problems in pure mathematics. Experimental mathematics is the outcome.”

We continuously had in front of our eyes a generic student wishing to know more about mathematical analysis at the beginning of his or her student life. We tried to offer paths from the standard knowledge of a student to modern and exciting topics in this way showing that a student from the first or second year is able to understand certain research problems.

The book has been divided into ten chapters and covers topics on sets and numbers, linear and metric spaces, sequences and series of numbers and func- tions, limits and continuity, differential and integral calculus of functions of one or several variables, constants (mainly *π,* but not only) and algorithms for finding them, the *W* –*Z* method of summation, and estimates of algorithms and of certain combinatorial problems. Many challenging exercises accompany the text. Most of them have been the subjects of different mathematical com- petitions during the last few years. In this respect we consider that there is an appropriate balance between what is traditionally called theory and exercises. The topics of the last two chapters bring the student closer to topics be- longing also to computer science. In this way it is shown that the frontier between “pure” mathematics and other related topics is more or less a matter of taste.

It is the proper moment and place to express our sincere gratitude to Professor Heiner Gonska of the University of Duisburg-Essen, Germany, giving us, among others, the opportunity of using all the facilities of his department and library.

Thanks are due to Professor Jonathan M. Borwein of Dalhousie University, Halifax, Nova Scotia, Canada, for his constant and warm friendship along the years and to Professor Karl Dilcher of Dalhousie University, Halifax, Nova Scotia, Canada for his firm support in the publication of this book. The author is also grateful to the editors of Springer-Verlag, New York, for very strong and constant support offered to us.

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Cluj-Napoca, December 2007 *Marian Muresan* Babes-Bolyai University

**1Sets and Numbers**

The aim of this chapter is to introduce several basic notions and results con- cerning sets and numbers.

**1.1 Sets**

**1.1.1 The concept of a set**

The basic notion of set theory which was first introduced by Cantor1 occurs constantly in our results. Hence it would be useful to discuss briefly some of the notions connected to it before studying the mathematical analysis.

We take the notion of a set as being already known. Roughly speaking, a *set*(*collection, class, family*) is any identifiable collection of objects of any sort.We identify a set by stating what its *members* (*elements*) are. The the- ory of sets has been described axiomatically in terms of the notion “member of ” ([82]).

We make no effort to built the complete theory of sets, but will appeal throughout to intuition and elementary logic. The so-called “naive” theory of sets is completely satisfactory for us ([74]).

We usually adhere to the following notational conventions. Elements of sets are denoted by small letters: *a, b, c,* ..., *x, y, z, α, β, γ,* ... . Sets are denoted by capital Roman letters: *A, B, C,* ..., *X, Y,* ... . Families of sets are denoted by capital script letters: *A, B, C,* ... .

A set is often defined by some property of its elements. We write *{x | P*(*x*)*}* (where *P*(*x*) is some proposition about *x*) to denote the set of all *x* such that *P*(*x*) is true. Here *|* is read “such that”.

If the object *x is* an element of the set *A,* we write *x ∈ A*; and *x /∈ A* means that this *x is not* in *A* or that *it does not belong* to *A.*

1 Georg Ferdinand Ludwig Philipp Cantor, 1845–1918.

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We write *∅* for the *empty* (*void*) set. It has no member at all. For any object *x, {x}* denotes the set whose only member is *x.* Then *x ∈ {x},* but *x* = *{x}.* Similarly, *{x*1*,x*2*,...,xn}* is the set whose elements are precisely *x*1*, x*2*,* ..., *xn.* We emphasize that the order of elements in a set is irrelevant and that *{x, x}* = *{x}.*

**Examples.** Some examples of sets are listed below.

(a) The set of natural numbers, N = *{*0*,*1*,*2*,*3*,...}.* (b) The set of nonzero natural numbers, N*∗* = *{*1*,*2*,*3*,...}.* (c) The set of integers, Z = *{*0*,±*1*,±*2*,±*3*,...}.* (d) The set of rational numbers, Q = *{p/q | p, q ∈* Z*, q* = 0*}*; *p* is the

*numerator* and *q* is the *denominator* of the fraction *p/q.* (e) The set of positive integers less than 7*.* (f) The set of Romanian cities having more than five million inhabitants. (g) The set *S* of vowels in English alphabet. *S* may be written as *S* =

*{a, e, i, o, u}* or *S* = *{x | x* is a vowel in English alphabet*}. △*

The first axiomatic approach of the natural number system was realized by Peano2 in [117]. His system is based on five axioms and a succession function satisfying

(a) There is a natural number 1*.* (b) Every natural number *a* has a successor, denoted *s*(*a*)*.* (c) There is no natural number whose successor is 1*.* (d) Distinct natural numbers have distinct successors; that is, *a* = *b* if and

only if *s*(*a*) = *s*(*b*)*.* (e) If a property is satisfied by 1 and also by the successor of every natural

number that possesses it, then it is satisfied by all natural numbers.

It is clear that from the above axioms we get the set *{*1*,*2*,...}.* Nowadays zero is considered a natural number because it results in a richer algebraic structure. The only differences are that instead of (a), (c), and (e) one con- siders

(a*′*) There is a natural number 0*.* (c*′*) There is no natural number whose successor is 0*.* (e*′*) If a property is satisfied by 0 and also by the successor of every natural number that possesses it, then it is satisfied by all natural numbers.

Let *A* and *B* be sets such that every element of *A* is an element of *B.* Then *A* is called a *subset* of *B* and we write *A ⊂ B* or *B ⊃ A.* In such a case we also say *B* is a *superset* of *A.* If *A ⊂ B* and *B ⊂ A,* we write *A* = *B. A* = *B* denies *A* = *B.* If *A ⊂ B* and *A* = *B,* we say *A* is a *proper subset of B* and is sometimes written as *A* ⊊ *B.* We remark that under this idea of equality of sets, the empty set is unique; that is, if *∅*1 and *∅*2 are any two empty sets, we have *∅*1 *⊂ ∅*2 and *∅*2 *⊂ ∅*1*.*

2 Giuseppe Peano, 1858–1932.

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Let *A* be a set. By *P*(*A*) we denote the family of subsets of *A.* Thus *P*(*∅*) = *{∅,{∅}}.* For *A* = *{*1*,*2*},* we have *P*(*A*) = *{∅,{*1*},{*2*},{*1*,*2*}}.*

It is clear that if *A* is not a subset of *B,* the following statement has to be true “There exists an element *x* such that *x ∈ A* and *x /∈ B*.”

**1.1.2 Operations on sets**

Let *A* and *B* be two sets. The set of elements that belong to *A* or to *B* (or to both) is called the *union of A and B,* and denoted *A∪B*; see Figure 1.1. We write

*A ∪ B* = *{x | x ∈ A* or *x ∈ B}.*

Let *A* be a family of sets. Then we write

*∪A* = *{x | x ∈ A* for some *A ∈ A}.*

Similarly, let *{Aα}α∈I* be a family of sets indexed by the index set *I.* We write

*∪α∈IAα* = *{x | x ∈ Aα* for some *α ∈ I}.* Let *A* and *B* be two sets. The set of elements that belong to both *A* and *B* is called the *intersection of A and B,* and denoted *A ∩ B*; see Figure 1.2. We write

*A ∩ B* = *{x | x ∈ A* and *x ∈ B}.*

Let *A* be a family of sets. Then we write

*∩A* = *{x | x ∈ A* for all *A ∈ A}.*

Similarly, let *{Aα}α∈I* be a family of sets indexed by the index set *I.* We write

*∩α∈IAα* = *{x | x ∈ Aα* for all *α ∈ I}.*

X

A B

A ∪ B

**Fig. 1.1.** Union

X

X

X

A B

A B

A B

A ∩ B

A ∩ B = ∅

A ∩ B = ∅

**Fig. 1.2.** Intersection

**Fig. 1.3.** Disjoint sets

**Fig. 1.3.** Disjoint sets

4 1 Sets and Numbers

**Theorem 1.1*.*** *Let A, B, and C be sets. Then*

(a) *A ∪ B* = *B ∪ A.* (a*′*) *A ∩ B* = *B ∩ A.* (b) *A ∪ A* = *A.* (b*′*) *A ∩ A* = *A.* (c) *A ∪ ∅* = *A.* (c*′*) *A ∩ ∅* = *∅.* (d) *A ∪* (*B ∪ C*)=(*A ∪ B*) *∪ C.* (d*′*) *A ∩* (*B ∩ C*)=(*A ∩ B*) *∩ C.* (e) *A ⊂ A ∪ B.* (e*′*) *A ∩ B ⊂ A.* (f) *A ⊂ B ⇐⇒ A ∪ B* = *B.* (f*′*) *A ⊂ B ⇐⇒ A ∩ B* = *A.*

Thus the union and intersection are commutative, associative, and idem- potent.

**Theorem 1.2*.*** *Let A, B, and C be sets. Then*

(a) *A ∩* (*B ∪ C*)=(*A ∩ B*) *∪* (*A ∩ C*)*, distributive law.* (b) *A ∪* (*B ∩ C*)=(*A ∪ B*) *∩* (*A ∪ C*)*, distributive law.*

**Theorem 1.3*.*** *Let X be a set and {Aα}α∈I a family of sets. Then*

(a) *X ∪* (*∪α∈IAα*) = *∪α∈I*(*X ∪ Aα*)*.* (b) *X ∩* (*∩α∈IAα*) = *∩α∈I*(*X ∩ Aα*)*.* (c) *X ∪* (*∩α∈IAα*) = *∩α∈I*(*X ∪ Aα*)*.* (d) *X ∩* (*∪α∈IAα*) = *∪α∈I*(*X ∩ Aα*)*.*

We say that *A* and *B* are *disjoint*, provided *A ∩ B* = *∅*; see Figure 1.3. Let *A* be a family of sets such that any two distinct members of *A* are disjoint. Then family *A* is said to be *pairwise disjoint*. Thus an indexed family *{Aα}α∈I* is pairwise disjoint if *Aα ∩ Aβ* = *∅* whenever *α, β ∈ I* and *α* = *β.*

A family *A* of nonempty subsets of a set *S* is said to be a *partition* of *S,* provided the following two conditions are satisfied.

(i) *S* = *∪A∈AA.* (ii) *A* is a pairwise disjoint family.

Thus each element in *S* belongs to one and only one set *A ∈ A.*

Let *A* and *B* be two sets. Then

*A \ B* = *{x | x ∈ A* and *x /∈ B}*

is said to be the *difference* of *A* and *B*; see Figure 1.4.

Let *A* be a subset of a set *X.* The *complement of A* (relative to *X* ) is the set *{x | x ∈ X, x /∈ A}.* This set is denoted by С*XA* or С*A* (when the set with respect to which the complement is considered is irrelevant). Other notation is *X \ A.*

**Proposition 1.1*.*** *We are given two sets X and A so that A ⊂ X. Then* С*X*(С*XA*) = *A.*

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X

A B

A \ B

X

A B

A∆B

**Fig. 1.4.** Difference of sets

**Fig. 1.5.** Symmetric difference

**Theorem 1.4*.*** (de Morgan3 laws) *The following equalities are true.*

(a) С(*A ∪ B*)=(С*A*) *∩* (С*B*)*.* (b) С(*A ∩ B*)=(С*A*) *∪* (С*B*)*.* (c) С(*∪α∈IAα*) = *∩α∈I*С*Aα.* (d) С(*∩α∈IAα*) = *∪α∈I*С*Aα.*

Let *A* and *B* be two sets. The *symmetric difference of A and B* is the set (*A \ B*) *∪* (*B \ A*) and we write *A△B* for this set; see Figure 1.5. Note that *A△B* is the set consisting of those elements which are in exactly one of *A* or *B.* It can be defined as well by

*A△B* = (*A ∩* С*B*) *∪* (С*A ∩ B*)*.*

Sometimes it becomes significant to consider the order of the elements in a set. If we consider a pair (*x*1*,x*2) of elements in which we distinguish *x*1 as the first element and *x*2 as the second element, then (*x*1*,x*2) is called an *ordered pair*. Thus, two ordered pairs (*x, y*) and (*u, v*) are equal if and only if *x* = *u* and *y* = *v.*

Let *X* and *Y* be nonempty sets. The *Cartesian* 4 *product of X and Y* (in this order) is the set *X × Y* of all ordered pairs (*x, y*) such that *x ∈ X* and *y ∈ Y.* Hence, *X × Y* = *{*(*x, y*) *| x ∈ X, y ∈ Y }.* Generally, *X × Y* = *Y × X.*

**Remark.** (1*,*2) = (2*,*1) and *{*1*,*2*}* = *{*2*,*1*}. △*

We usually write *X*2 instead of *X × X.* The Cartesian product of three sets, say *X,Y,* and *Z* (in this order) is defined as

(*X × Y* ) *× Z* = *X ×* (*Y × Z*) = *X ×Y × Z* = *{*(*x,y,z*) *| x ∈ X, y ∈ Y,z ∈ Z}.*

3 August de Morgan, 1806–1871. 4 Renatus Cartesius, the Latin name of René Du Perron Descartes, 1596–1650.

6 1 Sets and Numbers

Generally, we have

*X*1 *× X*2 *×···× Xn* = *{*(*x*1*,x*2*,...,xn*) *| x*1 *∈ X*1*, x*2 *∈ X*2*,...,xn ∈ Xn}.*

Thus*Xn* = *X* } *× X ×···×* {{ *X* }

*n* times

= *{*(*x*1*,x*2*,...,xn*) *| xi ∈ X, i* = 1*,*2*,...,n}.*

**Proposition 1.2*.*** *Let A, B, and C be sets. Then*

(a) *A ×* (*B ∪ C*)=(*A × B*) *∪* (*A × C*)*.* (b) *A ×* (*B ∩ C*)=(*A × B*) *∩* (*A × C*)*.* (c) *Suppose C and D are nonempty. Then C × D ⊂ A × B ⇐⇒ C ⊂ A*

*and D ⊂ B.* (d) *A × B* = *∅ ⇐⇒ A* = *∅ or B* = *∅.* (e) *A × B* = *B × A ⇐⇒ A* = *B.*

**1.1.3 Relations and functions**

A (*binary*) *relation R on two sets X and Y* is a subset of the Cartesian product of *X* and *Y* ; that is, *R* is a relation on *X* and *Y ⇐⇒ R ⊂ X ×Y.*

Let *R* be a relation on *X* and *Y.* The *domain of R* is the set

Dom*R* = *{x ∈ X |* (*x, y*) *∈ R* for some *y ∈ Y }.*

The *range of R* is the set

Range *R* = *{y ∈ Y |* (*x, y*) *∈ R* for some *x ∈ X}.*

The symbol *R−*1 denotes the *inverse of R*, i. e., *R−*1 = *{*(*y,x*) *|* (*x, y*) *∈ R}.* Let *R* and *Q* be relations. The *composition* (*product*) of two relations *R* and *Q* is the relation

*R ◦ Q* = *{*(*x, z*) *|* for some *y,* (*x, y*) *∈ Q* and (*y,z*) *∈ R}.*

The composition of *R* and *Q* may be empty. *R ◦ Q* = *∅ ⇐⇒* (Range*Q*) *∩* (Dom*R*) = *∅.* Given *R* a relation on *X* and *Y, A ⊂ X.* The *image of A under R* is the set

*R*(*A*) = *{y ∈ Y |* exists *x ∈ A* such that (*x, y*) *∈ R}.*

**Proposition 1.3*.*** *Let R, Q, and S be relations, and A and B be sets. Then*

(a) (*R−*1)*−*1 = *R.* (b) (*R ◦ Q*)*−*1 = *Q−*1 *◦ R−*1*.* (c) *R ◦* (*Q ◦ S*)=(*R ◦ Q*) *◦ S.* (d) (*R ◦ Q*)(*A*) = *R*(*Q*(*A*))*.* (e) *R*(*A ∪ B*) = *R*(*A*) *∪ R*(*B*)*.* (f) *R*(*A ∩ B*) *⊂ R*(*A*) *∩ R*(*B*)*.*

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An *equivalence relation* on a nonempty set *X* is a relation ∼ *⊂ X × X* such that for all *x, y,* and *z* in *X* the following conditions are satisfied.

(i) *x* ∼ *x* (reflexive). (ii) *x* ∼ *y* implies *y* ∼ *x* (symmetric). (iii) *x* ∼ *y* and *y* ∼ *z* imply *x* ∼ *z* (transitive).

**Examples.** (a) The usual “=” on Q is an equivalence relation on Q*.* (b) Let Z be the set of integers and settle down a natural number *n.* For every *a, b ∈* Z*,* we say “*a is congruent to b* modulo *n*” if *a−b* = *kn* for some integer *k.* Here “congruence modulo *n*” is an equivalence relation on Z*.* In notational form we write

*a* = *b* (mod *n*) *⇐⇒ ∃ k ∈* Z such that *a − b* = *kn.*

(c) Let Z be the set of integers and *x, y ∈* Z*.* Define *x* ∼ *y* if and only if *x − y* is even. It is easy to check that ∼ is an equivalence relation on Z*. △*

**Proposition 1.4*.*** *Suppose* ∼ *is an equivalence relation on nonempty set X. Then it defines a partition of X. Conversely, each partition of X defines an equivalence on X.*

*Proof.* Define a nonempty subset *S* of *X* by

*x, y ∈ S ⇐⇒ x* ∼ *y.*

If *x ∈ X,* then *S* = *∅.* Obviously, the class of sets *{S}* is a partition of *X.*

Conversely, let *{S}* be a partition of *X.* Define an equivalence relation on *X* by

*x* ∼ *y ⇐⇒ x, y ∈ S. 2*

The partition of *X* generated by an equivalence relation ∼ on *X* is de- noted *X/* ∼ and is said to be the *equivalence classes* generated by ∼ *.*

Let *P* be a nonempty set. A *partial ordering* on *P* is a relation *≤ ⊂ P ×P* such that for each *x, y,* and *z* in *P* one has

(i) *x ≤ x* (reflexive). (ii) *x ≤ y* and *y ≤ x* imply *y* = *x* (antisymmetric). (iii) *x ≤ y* and *y ≤ z* imply *x ≤ z* (transitive).

Assume *≤* is a partial ordering on *P.* Then the pair (*P,≤*) is called a *partially ordered set*.

**Examples 1.1*.*** (a) Let *X* be a nonempty set and consider *A, B ⊂ X.* Define *A ≤ B* whenever *A ⊂ B.* Then “ *≤* ” is a partial ordering on the class of subsets of *X.* (b) For *m, n ∈* N define *m ≤ n* if there exists *k ∈* N*∗* such that *m* = *kn.* Then “ *≤* ” is a partial ordering on N*. △*

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If, moreover, the partial ordering relation *≤* satisfies

(iv) *x, y ∈ P* implies *x ≤ y* or *y ≤ x,*

then *≤* is called a *total ordering on P.* Assume *≤* is a total ordering on *P.* Then the pair (*P,≤*) is called a *totally ordered set*.

If *x ≤ y* and *x* = *y,* then we write *x < y.* The expression *x ≥ y* means *y ≤ x* and *x>y* means *y < x.*

Law (iv) is sometimes stated “for arbitrary elements *x* and *y* in a totally ordered set exactly one of the relations *x<y, x* = *y, x>y* are true.” This law is called the *trichotomy law*.

**Examples.** (a) The usual *≤,* meaning “less or equal”, is a total ordering on Q*.* (b) Let *A* be a nonempty set. Then the relation *⊂* on the class of all subsets of *A* is a partial ordering on it, as we already saw by (a) of Examples 1.1. We emphasize it is not a total ordering on *A. △*

A subset *Q* of a partially ordered set *P* with order *≤* is said to be a *down-set* of *P* if whenever *x ∈ Q, y ∈ P,* and *y ≤ x,* then *y ∈ Q.* Let *R* be an arbitrary subset of a partially ordered set *P* with order *≤ .* Then the smallest down-set containing *R,* denoted *↓ R*is the set of all *x ∈ P* for which there is a *y ∈ R* such that *x ≤ y.* If *R* is a singleton (i.e., *R* = *{r}*), then *↓ R* =*↓ r* is said to be a *principal down-set* of *P.*

Suppose *≤* is a total ordering on *P* such that

(v) *∅ ̸*= *A ⊂ P* implies there exists an element *a ∈ A* such that *a ≤ x* for

each *x ∈ A* (*a* is the *smallest element of A*);

then *≤* is called a *well ordering on P.*

Assume *≤* is a well ordering on *P.* Then the pair (*P,≤*) is called a *well- ordered set*.

**Example.** The set N of natural numbers with the usual ordering *≤* is a well-ordered set, whereas Z with the usual ordering *≤* is not a well-ordered set. *△*

A binary relation *≼* on a nonempty set *X* that is only transitive and reflexive is said to be a *quasi-order on X.* Then the *symmetric core ≡* of *≼,* defined by

*x ≡ y ⇐⇒ x ≼ y* and *y ≼ x,*

is an equivalence relation on *X.* Moreover, *≼* defines a relation *⊑* on the quotient set *X/ ≡* of equivalence classes *⌈x*] = *{y ∈ X | x ≡ y}* as

*⌈x*] *⊑ ⌈y*] *⇐⇒ x ≼ y.* (1.1)

Note, the relation *⊑* is a partial order on *X/ ≡ .* Conversely

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**Proposition 1.5*.*** *Given an equivalence relation ≡ on X together with a partial order ⊑ on the equivalence classes X/ ≡*= *{⌈x*] *| x ∈ X}, condition* (1.1) *defines a quasi-order ≼ on X.*

Let *P* be a totally ordered set. For *x, y ∈ P,* we define *y* = max*{x, y}* if *x ≤ y,* and *x* = max*{x, y}* if *y ≤ x.* For a finite subset *{x*1*,...,xn}* (not all *xk*s are necessarily distinct), we define max*{x*1*,...,xn}* = max*{*max*{x*1*, ...,xn−*1*},xn}.* Similarly, we define min*{x, y}.* That is, *x* = min*{x, y}* when- ever *x ≤ y,* and *y* = min*{x, y}* whenever *y ≤ x.* Also min*{x*1*,...,xn}* = min*{*min*{x*1*,..., xn−*1*},xn}.*

Let (*P,≤*) be a partially ordered set and *A* be a nonempty subset of *P.* An element *x ∈ P* is said to be

(i) *A lower bound of A* if *x ≤ y* for every *y ∈ A.* In this case we say that *A*

is *bounded below*. (ii) *An upper bound of A* if *y ≤ x* for every *y ∈ A.* In this case we say that

*A* is *bounded above*. (iii) The *greatest lower bound of A* or *infimum of A* if

(iii1) *x* is a lower bound of *A,* (iii2) If *x < y,* then *y* is not a lower bound of *A.* (iv) The *least upper bound of A* or *supremum of A* if

(iv1) *x* is an upper bound of *A,* (iv2) If *y < x,* then *y* is not an upper bound of *A.* A nonempty subset *A* of a partially ordered set is said to be *bounded* if it is bounded below and above. *A* is said to be *unbounded* if it is not bounded.

**Remark 1.1*.*** A nonempty subset *A* of a partially ordered set may have sev- eral lower and/or upper bounds whereas it has at most one infimum (denoted inf *A*) and at most one supremum (denoted sup *A*). *△*

**Example.** Let *A* consist of all numbers 1*/n,* where *n* = 1*,*2*,....* Then by the usual *≤* on the set of rational numbers the set *A* is bounded, sup*A* = 1*,* inf *A* = 0*,* and 1 *∈ A* whereas 0 */∈ A. △*

Let *f* be a relation and *A* be a set. As we already saw the image of *A* under *f* is the set *f*(*A*) = *{y |* (*x, y*) *∈ f* for some *x ∈ A}*. Note *f*(*A*) = *∅ ⇐⇒ A∩*Dom*f* = *∅.* If *f*(*A*) *⊂ B,* then this is interpreted as “*f* maps the set *A* into the set *B.*” The *inverse image of A under f* is the set

*f −*1(*A*) = *{x |* (*x, y*) *∈ f* for some *y ∈ A}* = *{x | f*(*x*) *∩ A* = *∅}.*

A relation *f* is said to be *single-valued* if (*x, y*) *∈ f* and (*x, z*) *∈ f* imply *y* = *z.* In such a case we write *f*(*x*) = *y.* Otherwise it is also called a *multifunction* or *set-valued function* or even *correspondence*. A single-valued relation is said to be a *function* (*mapping*, *map*, *application*, *transformation*, *operator*). If *f* and *f −*1 are both single-valued, *f* is said to be a *bijective* function, Figure 1.8.

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f a

α

bcγ**Fig. 1.6.** Non-one-to-one

f

f

**Fig. 1.7.** Non-onto

**Fig. 1.8.** Bijective

**Theorem 1.5*.*** *Let X and Y be nonempty sets and f ⊂ X ×Y be a relation. Suppose that {Ai}i∈I is a family of subsets of X and {Bj}j∈J is a family of subsets of Y. Then*

(a) (b) *f*(*∪f −*1*i∈IA*(*∪j∈JBi*) = *j*) *∪*= *i∈If*(*A∪j∈Jfi*)*.*

*−*1(*Bj*)*.* (c) *f*(*∩i∈IAi*) *⊂ ∩i∈If*(*Ai*)*. The following statements are true if f is a function, but may fail for arbitrary relations.* (d) (e) *f f −*1*−*1a

α

a

α

bβbβγcγ(*∩*(С*Y j∈JBB*) = *j*) С= *X*(*f ∩j∈Jf−*1(*B*))*, −*1(*BB j*)*. ⊂ Y.* (f) *f*(*f −*1(*B*) *∩ A*) = *B ∩ f*(*A*)*, A ⊂ X, B ⊂ Y.*

Let *f* be a function such that Dom*f* = *X* and Range*f ⊂ Y.* Then *f* is said to be a function *from* (on) *X* into (to) *Y* and we write *f* : *X → Y.* If Range *f* = *Y,* we say that *f* is *onto*, or *surjective;* that is, *f*(*X*) = *Y.* It means that for every *y ∈ Y* there exists an *x ∈ X* such that *y* = *f*(*x*)*.*

*f* is said to be *one-to-one* or *injective* if for any *x, t ∈ X* with *x* = *t,* one has *f*(*x*) = *f*(*t*)*.* Equivalently, *f*(*x*) = *f*(*t*) implies *x* = *t.* In other words, a function *f* : *X → Y* is said to be one-to-one if distinct elements in *X* have distinct images in *Y,* that is, if no two different elements in *X* have the same image. Figure 1.6 exhibits a surjective but not a one-to-one function, Figure 1.7 exhibits a one-to-one but not an onto function, and Figure 1.8 presents a bijective function.

**Theorem 1.6*.*** *Let X and Y be nonempty sets and f* : *X → Y be a function. Then f is bijective if and only if it is one-to-one and onto.*

Consider *f* : *X → Y* and *∅ ̸*= *A ⊂ X.* A function *fA* : *A → Y* defined as *fA*(*x*) = *f*(*x*)*,* for all *x ∈ A,* is said to be the *restriction* of *f* to *A.*

Consider *f* : *A → Y* and a set *X ⊃ A.* A function *g* : *X → Y* satisfying *g*(*a*) = *f*(*a*)*,* for every *a ∈ A,* is said to be an *extension* of *f* to *X.*

Let *X* be a nonempty set. A *sequence* in *X* is a function having N*∗* or N or even an unbounded subset of it as its domain and *X* its range. Let *x* be such a function. We often write *xn* instead of *x*(*n*) for the value of *x* at *n.* The value *xn* is called the *n*th *term* of the sequence. The sequence whose

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*n*th term is *xn* is denoted by (*xn*)*∞n*=1*,* (*xn*)*n≥*1*,* (*xn*)*n,* or (*xn*)*.* A sequence (*xn*) is said to be *in* a set *X* if *xn ∈ X* for each *n.* We distinguish between the sequence (*xn*) (that is a function) and the set *{xn}* (that is the range of the function). We note that (*xn*) is an ordered set, whereas *{xn}* is not. In most of the cases we are interested in the behavior of sequences an *n* increases without bound, that is, in their limiting behavior.

**1.2 Sets of numbers**

A satisfactory discussion of the main concepts of analysis (e.g., convergence, continuity, differentiation, and integration) have to be based on an accurately defined number concept.

We do not, however, enter into any discussion of the axioms governing the arithmetic of the integers, but we take the rational number system as our starting point.

**1.2.1 Two examples**

It is well known that the rational number system is inadequate for many purposes. Maybe the oldest and the most frustrating case is the following. Consider an isosceles right triangle having the length of the catheti equal to 1*.* Can we express the length of the hypotenuse by a rational number? By the Pythagoras 5 theorem the square of the hypotenuse is 2*.*

**Example 2.1*.*** We start by showing that the equation

*p*2 = 2 (1.2)

is not satisfied by any rational *p.* For, suppose that (1.2) is satisfied. Then we can write *p* = *m/n,* where *m* and *n* are integers with *n* = 0*.* We may assume that *m* and *n* have no common divisor. Then (1.2) implies

*m*2 = 2*n*2*.* (1.3)

This shows that *m*2 is even. Hence *m* is even (if *m* is odd, *m*2 is odd as well), and so *m*2 is divisible by 4*.* It follows that the right-hand side of (1.3) is divisible by 4*,* so that *n*2 is even, which implies that *n* is even.

Thus the assumption that (1.2) holds for a rational number leads us to the conclusion that both *m* and *n* are even, contrary to our choice on *m* and *n.* Hence (1.2) is impossible for rational *p.* So, the length of the hypotenuse to an isosceles right triangle with unitary catheti is nonrational. Several comments on this topic may be found at page 447.

We get the same conclusion following a different path. Relation (1.3) can- not have a nonzero solution in integers because the last nonzero digit of a

5 Pythagoras of Samos (*Πυθαγ* ́*oρας o Σ* ́*αμιoς*)*, ∼*560–*∼*480 (BC).

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square, written in base 3*,* is necessarily 1*,* whereas the last digit of twice a square is 2*. △*

We examine this situation a little more closely.

**Example 2.2*.*** Let *A* be the set of all positive rationals *p* such that *p*2 *<* 2*,* and let *B* be the set of all positive rationals *p* such that *p*2 *>* 2*. A* and *B* are nonempty because 1 *∈ A* and 2 *∈ B.* We show that *A* contains no largest element, and *B* contains no smallest element.

More explicitly, for every *p ∈ A* we can find a rational *q ∈ A* such that *p < q,* and for every *p ∈ B* we can find a rational *q ∈ B* such that *q < p.*

Suppose that *p ∈ A.* Then *p*2 *<* 2*.* Choose a rational *h* such that 0 *< h <* 1 and such that

*h <* 2 *− p*2

2*p* + 1 *.* Put *q* = *p* + *h.* Then *q > p,* and

*q*2 = *p*2 + (2*p* + *h*)*h<p*2 + (2*p* + 1)*h<p*2 + (2 *− p*2)=2*,*

so that *q* is in *A.* This proves the first part of our assertion.

Next, suppose that *p ∈ B.* Then *p*2 *>* 2*.* Put

*q* = *p −* (*p*2 *−* 2)*/*2*p* = *p/*2+1*/p.*

Then 0 *<q<p* and

*q*2 = *p*2 *−* (*p*2 *−* 2) +

(*p*2 *−* 2

2*p*

)2

*> p*2 *−* (*p*2 *−* 2) = 2*,*

so that *q ∈ B. △*

**Remark.** The purpose of the above discussion was to show that the rational number system has certain gaps, in spite of the fact that between any two distinct rationals there is another one (because *p <* (*p* + *q*)*/*2 *< q*). A deeper result is introduced by Corollary 2.11. *△*

**1.2.2 The real number system**

There are several ways to introduce the real number set. We selected one of them assuming that it is easiest and shortest.

We say that a set *X* is the *real number set* provided on it there are defined two operations

*X × X ∋* (*x, y*) *↦→ x* + *y ∈ X, X × X ∋* (*x, y*) *↦→ xy ∈ X*

called *addition* and *multiplication* as well as a binary relation *≤* called *order- ing* and satisfying the following axioms (conditions, assumptions).

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(R1) (*x* + *y*) + *z* = *x* + (*y* + *z*)*, ∀x,y,z ∈ X.* (R2) There exists an element 0 *∈ X,* called *zero* or *null* such that *x* +0=

0 + *x* = *x, ∀x ∈ X.* (R3) For each *x ∈ X* there exists an element *−x ∈ X,* called the *opposite*

to *x,* such that *x* + (*−x*)=(*−x*) + *x* = 0*.* (R4) *x* + *y* = *y* + *x, ∀x, y ∈ X.* (R5) (*xy*)*z* = *x*(*yz*)*, ∀x,y,z ∈ X.* (R6) There exists an element 1 *∈ X \ {*0*},* called *unity* or *identity*, such

that *x ·* 1=1 *· x* = *x, ∀x ∈ X.* (R7) For each element *x ∈ X\{*0*},* there exists an element *x−*1 *∈ X,* called

the *inverse* of *x,* such that *xx−*1 = *x−*1*x* = 1*.* (R8) *xy* = *yx, ∀x, y ∈ X.* (R9) *x*(*y* + *z*) = *xy* + *xz, ∀x,y,z ∈ X.* (R10) *x ≤ x, ∀x ∈ X.* (R11) For every *x, y ∈ X, x ≤ y* and *y ≤ x* imply *x* = *y.* (R12) For every *x,y,z ∈ X, x ≤ y* and *y ≤ z* imply *x ≤ z.* (R13) For every *x, y ∈ X* we have *x ≤ y* or *y ≤ x.* (R14) For every *x,y,z ∈ X, x ≤ y* implies *x* + *z ≤ y* + *z.* (R15) For every *x, y ∈ X, x ≥* 0 and *y ≥* 0*,* imply *xy ≥* 0*.* (R16) For every ordered pair (*A, B*) of nonempty subsets of *X* having the property that *x ≤ y* for every *x ∈ A* and *y ∈ B* there exists an element *z ∈ X* such that

*x ≤ z ≤ y,* for every *x ∈ A* and *y ∈ B.*

**Remarks.** From (R1)–(R4) we have that (*X,*+) is an *Abelian* 6 (*commuta- tive*) group. So the null element and the opposite of an element are unique. Also *−*0 = 0 and *−*(*−x*) = *x,* for all *x ∈ X.* From (R5)–(R8) we have that (*X \ {*0*},·*) is an Abelian group. Therefore the identity element and the inverse element of an element are unique. Also 1*−*1 = 1 and (*x−*1)*−*1 = *x,* for all *x ∈ X \ {*0*}.* From (R1)–(R9) we have that (*X,*+*,·*) is a *field*. From (R10)–(R13) it follows that (*X,≤*) is a *totally ordered* set. From (R1)–(R15) one has that (*X,*+*,·,≤*) is a *totally ordered field*. Axioms (R14) and (R15) ex- press the compatibility of the ordering relation with the algebraic operations. Axiom (R16) has a special rôle that is made clear later.

We see at once that the set of rational numbers Q is a totally ordered field. At the same time the set of rational numbers does not fulfill Axiom (R16), as we already saw by Example 2.2. The element *x−*1 *∈ X \ {*0*}* from (R7) is denoted as 1*/x,* too. Hence *y/x* = *yx−*1*. △*

For a while we ignore assumption (R16).

**Proposition 2.6*.*** *There hold x ·* 0=0 *· x* = 0*.*

6 Niels Henrik Abel, 1802–1829.

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*Proof.* We have

*x ·* 0 = *x*(0 + 0) = *x ·* 0 + *x ·* 0 =*⇒ x ·* 0 = *x ·* 0 *− x ·* 0=0*.*

Similarly, 0 *· x* = 0*. ⊓⊔*

**Remarks.** From Proposition 2.6 it follows that 0 has no inverse because for each *x ∈ X,* 0 *· x* = 0 and 0 = 1 *∈ X \ {*0*}.* It also follows that the multiplication (defined on *X* and associative and commutative on *X \ {*0*}*) is associative and commutative on *X.* Obviously, if *x* = 0 or *y* = 0*,* then *x · y* = 0*. △*

**Proposition 2.7*.*** *If xy* = 0*, then x* = 0 *or y* = 0*.*

*Proof.* Suppose *x* = 0*.* Then there exists *x−*1*.* From one side *x−*1(*xy*) = (*x−*1*x*)*y* = *y* and from the other side *x−*1(*xy*) = *x−*10=0*.* Thus if *x* = 0*,* then *y* = 0*.* Similarly, by commutativity, if *y* = 0*,* then *x* = 0*. ⊓⊔*

**Proposition 2.8*.*** *For every x ∈ X, −x* = (*−*1) *· x.*

*Proof.* We have *x*(1 + (*−*1)) = *x ·* 0 = 0 and *x*(1 + (*−*1)) = *x ·* 1 + *x ·* (*−*1) = *x* + *x ·* (*−*1)*.* Then the conclusion follows. *⊓⊔*

**Corollary 2.1*.*** *One has* (*−*1)2 = (*−*1)(*−*1) = 1*, x*(*−y*)=(*−x*)*y* = *−*(*xy*)*, for all x, y ∈ X.*

*Proof.* (*−*1)2 = (*−*1)(*−*1) = *−*(*−*1) = 1*. x*(*−y*) = *x*(*−*1)*y* = (*−*1)*xy* = *−*(*xy*) and (*−x*)*y* = (*−*1)*xy* = *−*(*xy*)*. ⊓⊔*

**Proposition 2.9*.*** *Consider x,y,z,xi, and yi belonging to X, i* = 1*,*2*. Then based on* (R1)–(R15) *one has*

(1) (a) *x*1 *≤ x*2 and *y*1 *≤ y*2 imply *x*1 + *y*1 *≤ x*2 + *y*2*.* (b) *x*1 *< x*2 and *y*1 *≤ y*2 imply *x*1 + *y*1 *< x*2 + *y*2*.* (2) (a) *x >* 0 if and only if *x−*1 *>* 0*.*

(b) *x ≥* 0 implies *− x ≤* 0*.* (c) *x >* 0 implies *− x <* 0*.* (3) (a) *x ≤ y* and *z >* 0 imply *xz ≤ yz.* (b) *x<y* and *z >* 0 imply *xz < yz.* (c) *x ≤ y* and *z <* 0 imply *xz ≥ yz.* (d) *x<y* and *z <* 0 imply *xz > yz.* (4) If *xy >* 0*,* then *x ≤ y* if and only if 1*/x ≥* 1*/y.* (5) (a) 0 *≤ x*1 *≤ x*2 and 0 *≤ y*1 *≤ y*2 imply *x*1*y*1 *≤ x*2*y*2*.* (b) 0 *< x*1 *< x*2 and 0 *< y*1 *≤ y*2 imply *x*1*y*1 *< x*2*y*2*.* (c) *x*1 *≤ x*2 *≤* 0 and *y*1 *≤ y*2 *≤* 0 imply *x*1*y*1 *≥ x*2*y*2*.* (d) *x*1 *< x*2 *≤* 0 and *y*1 *≤ y*2 *<* 0 imply *x*1*y*1 *> x*2*y*2*.*

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The *absolute value function* is defined as

for *x ∈ X, |x|* =

{*x, x ≥* 0*, −x, x <* 0*.*

So, *|·|* : *X →* [0*,∞*[*.*

**Proposition 2.10*.*** *From* (R1)–(R15) *follow that for every x, y ∈ X, we have*

(1) (a) *|x| ≥* 0*.*

(b) *|x|* = 0 *⇐⇒ x* = 0*.* (c) *|x|* = *| − x|.* (2) (a) *|x* + *y|≤|x|* + *|y|.*

(b) *|x − y|≥||x|−|y||.* (3) (a) *|x| ≤ a ⇐⇒ −a ≤ x ≤ a.* (b) *|x| < a ⇐⇒ −a<x<a.* (4) (a) *|xy|* = *|x|·|y|.*

(b)

∣∣∣∣*xy*∣∣∣∣ = *|x| |y|.* (c) *|xn|* = *|x|n, n ∈* N*∗.*

The *distance function* is defined as

for *x, y ∈ X, d*(*x, y*) = *|x − y|.*

Thus *d* : *X × X →* [0*,∞*[*.*

**Proposition 2.11*.*** *From Proposition 2.10 it follows that*

*d*(*x, y*)=0 *if and only if x* = *y*; *d*(*x, y*) = *d*(*y,x*)*, for all x, y ∈ X*; *d*(*x, y*) *≤ d*(*x, z*) + *d*(*z,y*)*, for all x,y,z ∈ X.*

The *signum function* is defined as

for *x ∈ X,* sign*x* =

1*, x >* 0*,* 0*, −*1*, x* = 0*, x <* 0*.*

Therefore sign : *X → {−*1*,*0*,*1*}.*

**Warning.** There exist several systems satisfying (R1)–(R16) axioms. But all are algebraically and order isomorphic, [77, Theorem 5.34], [112, vol.1, *§* 2.9]. We choose one of them and call it *the set of real numbers*, and denoted it by

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R = (R*,*+*,·,≤*). An element in R is said to be a *real number*. A real number *x* such that 0 *≤ x* is said to be *nonnegative*, whereas if 0 *< x,* it is called a *positive* number. A real number *x* such 0 *≥ x* is said to be *nonpositive*, whereas if 0 *> x,* it is called a *negative* number.

**Proposition 2.12*.*** *Number* 1 *is positive.*

*Proof.* Suppose that 1 is nonpositive; that is, 1 *≤* 0*.* Adding *−*1 to both sides we have 0 *≤ −*1*.* Multiplying both sides by the nonnegative number *−*1 and using (R15), we get 0 *≤* (*−*1)(*−*1) *⇐⇒* 0 *≤* 1*.* Now, 1 is simultaneously nonnegative and nonpositive, thus 0 = 1*.* But this contradicts (R6). Hence 1 *>* 0*. ⊓⊔*

It is clear that any set of real numbers (i.e., any subset of R) having an infimum is nonempty and bounded below. The converse statement is also true.

**Theorem 2.1*.*** *Every nonempty and bounded below subset A of* R *has an infimum.*

*Proof.* Denote by *A*0 the set of lower bounds of *A.* Because *A* is bounded below, *A*0 = *∅.* Remark that the ordered system (*A*0*,A*) has the property that for every *x ∈ A*0 and *y ∈ A* it holds that *x ≤ y.* From (R16) it follows there exists a real number *z* such that

*x ≤ z ≤ y,* for all *x ∈ A*0 and *y ∈ A.*

It results that number *z* is the greatest element in *A*0*,* that is, an infimum of *A.* By Remark 1.1 we conclude that *z* is the infimum of *A. ⊓⊔*

**Corollary 2.2*.*** *If A is a nonempty and bounded below subset of* R *and B is a nonempty subset of A, then*

inf *A ≤* inf *B.*

**Theorem 2.2*.*** *Every nonempty and bounded above subset A of* R *has a supremum.*

**Corollary 2.3*.*** *If A is a nonempty and bounded above subset of* R *and B is a nonempty subset of A, then*

sup *A ≥* sup *B.*

**Remark.** The proofs of the existence of an infimum and the existence of a supremum have used Axiom (R16). At the same time it can be proved (and we see immediately) that (R16) follows from any one of these theorems. Thus (R16) is equivalent to any one of these theorems. Hence we may substi- tute (R16) by one of these statements in order to get the same real number system. *△*

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**Theorem 2.3*.*** *Suppose X is a totally ordered field (i.e., it satisfies axioms* (R1)–(R15)) *and, moreover, every nonempty and bounded above subset of it has a supremum. Then Axiom* (R16) *is fulfilled. Proof.* Consider an ordered pair (*A, B*) of nonempty subsets of *X* having the property that *x ≤ y* for any *x ∈ A* and *y ∈ B.* Then *A* is nonempty and bounded above (by any element of *B*). It follows that there exists *z ∈ X* such that

*z* = sup*A.* (1.4) We have to show that *z ≤ y,* for every *y ∈ B.* For, suppose there exists *y*0 *∈ B* such that *y*0 *< z.* Then *y*0 is an upper bound of *A* strictly less then *z,* contradicting assumption (1.4). *⊓⊔*

A similar statement is true for the infimum.

**Theorem 2.4*.*** *Suppose X is a totally ordered field and, moreover, every nonempty and bounded below subset of it has an infimum. Then Axiom* (R16) *is fulfilled.*

**Theorem 2.5*.*** (Archimedes’ 7 principle) *For every two real numbers x and y such that y >* 0 *there exists a natural number n such that x < ny.*

*Proof.* Under the above-mentioned assumptions define

*A* = *{u ∈* R *|* exists *n ∈* N*∗* with *u < ny}.*

and remark that *A* = *∅* (because at least *y ∈ A*). We show that *A* = R*.* Suppose that *A* = R and denote *B* = R *\ A.* Obviously, *B* = *∅.*

Note that for every *u ∈ A* and *v ∈ B, u < v.* Indeed, for every *u ∈ A* there exists a natural *n* such that *u < ny.* Because *v /∈ A* and the real number set is a totally ordered set, it follows that *ny ≤ v.* Then

*u < ny ≤ v* =*⇒ u < v.*

Axiom (R16) implies that for the ordered pair (*A, B*) there exists a real number *z* such that

*u ≤ z ≤ v,* for all *u ∈ A, v ∈ B.* (1.5)

The real number *z − y* belongs to *A,* because otherwise *z − y ∈ B,* and then by (1.5)

*z ≤ z − y* =*⇒ y ≤* 0*,* contradicting the hypothesis. Therefore *z−y ∈ A.* Then we can find a natural number *n* such that *z − y < ny.* We also have

*z* + *y* = (*z − y*)+2*y <* (*n* + 2)*y,*

and it follows that *z* +*y ∈ A.* Then *z* +*y ≤ z,* thus *y ≤* 0*.* The contradiction shows that *A* = R and the theorem is proved. *⊓⊔*

7 Archimedes of Siracusa (*′Aρχιμ* ́*ηδης o Συρακo* ́*υσιoς*)*,* 287–212 (BC).

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**Corollary 2.4*.*** *Given a positive ε, there exists a natural number n so that* 1*/n < ε.*

**Corollary 2.5*.*** *The set of natural numbers is unbounded.*

**Theorem 2.6*.*** *The supremum of a nonempty and bounded above set is unique.*

*Proof.* Let *A* be the set under discussion. Then by Theorem 2.2 we know that there exists sup*A.* Suppose that sup*A* = *a*1 and sup*A* = *a*2 and *a*1 = *a*2*.* Then either *a*1 *< a*2 or *a*2 *< a*1*.* In both cases we get a contradiction.

We may argue equally well by Remark 1.1. *⊓⊔*

The following characterization of a supremum is useful.

**Theorem 2.7*.*** *A real number a is the supremum of a set A ⊂* R *if and only if*(i) *For every x ∈ A, x ≤ a.* (ii) *For every ε >* 0 *there is an element y ∈ A such that y>a − ε.*

*Proof.* (i) says that *a* is an upper bound of *A,* and (ii) shows that there is no upper bound less then *a. ⊓⊔*

Similar results hold in the case of an infimum.

**Theorem 2.8*.*** *The infimum of a nonempty and bounded below set is unique.*

**Theorem 2.9*.*** *A real number a is the infimum of a set A ⊂* R *if and only if*

(i) *For every x ∈ A, x ≥ a.* (ii) *For every ε >* 0 *there is an element y ∈ A such that y<a* + *ε.*

**Theorem 2.10*.*** *For every real x >* 0 *and every integer n ≥* 1*, there is one and only one real y >* 0 *such that yn* = *x.* **Remark.** This number *y* is written as *n√x* or *x*1*/n* and it is called the *n*th *root* or *radical* (of index *n*) of the positive real number *x.* The second root is called the *square root,* and the third root is called the *cube root*. *△*

*Proof.* If *n* = 1*, y* is precisely *x.* Suppose that *n ∈ {*2*,*3*,...}.*

That there is at most one such *y* is clear, because 0 *< y*1 *< y*2 implies *yn*1 *< yn*2*.*

Let *E* be the set consisting of all positive reals *t* such that *tn < x.* If *t* = *x/*(1 + *x*)*,* then 0 *<t<* 1; hence *tn <t<x,* so *E* is not empty. Put *t*0 =1+ *x.* Then *t*0 *>* 1 implies *tn*0 *> t*0 *> x,* so that *t*0 */∈ E,* and *t*0 is an upper bound of *E.* Let *y* = sup*E* (which exists, by Theorem 2.2).

Suppose *yn < x.* Choose *h* such that 0 *<h<* 1 and

*h < x − yn*

(1 + *y*)*n − yn.*

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We have

(*y* + *h*)*n* = *yn* +

(*n*1)*yn−*1*h* + *···* + *hn ≤ yn* + *h*((*n*1)) *yn−*1 + *···* + 1= *yn* + *h*((1 + *y*)*n − yn*) *< yn* + (*x − yn*) = *x.*

Thus *y* + *h ∈ E,* contradicting the fact that *y* is an upper bound of *E.*

Suppose *yn > x.* Choose *k* such that 0 *<k<* 1*, k<y,* and

*k <* (1 + *yn y*)*− n x*

*− yn.*

Then, for *t ≥ y − k,* we have

*tn ≥* (*y − k*)*n* = *yn −*

(*n*1)*yn−*1*k* +

(*n*2)*yn−*2*k*2 *−···* + (*−*1)*nkn*

= *yn − k*

((*n*1)*yn−*1 *−*

(*n*2)*yn−*2*k* + *···* + (*−*1)*n−*1*kn−*1)

*≥ yn − k*

((*n*1)*yn−*1 +

(*n*2)) *yn−*2 + *···* + 1= *yn − k*[(1 + *y*)*n − yn*] *> yn* + (*x − yn*) = *x.*

Thus *y − k* is an upper bound of *E,* contradicting the fact that *y* = sup*E.*

It follows that *yn* = *x. ⊓⊔*

a b [a, b] = {x ∈ R | a ≤ x ≤ b} closed interval [ ] a [a, b[= {x ∈ R | a ≤ x<b} left closed right open interval [ b [ a ] a, b] = {x ∈ R | a<x ≤ b} left open right closed interval ] b ] ]a, b[= {x ∈ R | a<x<b} open interval a ] b [

**Fig. 1.9.** Bounded intervals

An *interval A* of the real number system is a subset of R so that for every *x, y ∈ A* and *z ∈* R satisfying *x ≤ z ≤ y,* we have *z ∈ A.* An interval bounded below and above is said to be *bounded*. Otherwise it is called *unbounded*. For any nonempty and bounded interval *A,* the nonnegative real number *l*(*A*) = sup *A −* inf *A* is said to be the *length* of *A.*

We remark that for any real numbers *a* and *b* with *a ≤ b* the bounded intervals are listed in Figure 1.9.

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**Theorem 2.11*.*** *Let* (*Ik*)*k∈*N *be a nested sequence of nonempty closed and bounded intervals in* R*, that is,*

*Ik*+1 *⊂ Ik, k ∈* N*.* (1.6)

*Then*

*∩k∈*N*Ik* = *∅.*

*Proof.* Denote *Ik* = [*ak,bk*]*, k ∈* N*.* From (1.6) it follows that

*ak ≤ ak*+1 *≤ bk*+1 *≤ bk, k ∈* N*.* (1.7)

Denote *A* = *{x | x* = *ak,* for some *k ∈* N*}* and *B* = *{y | y* = *bk,* for some *k ∈* N*}.* Then for every *x ∈ A* and every *y ∈ B* we have *x ≤ y,* because otherwise there exist *ak ∈ A* and *bm ∈ B* such that

*bm < ak.*

We have either *m<k* or *k < m.* Suppose *m < k.* Then

*bm < ak ≤ bk,*

thus contradicting (1.7).

Axiom (R16) supplies a real *z* such that

*ak ≤ z ≤ bk, k ∈* N*.*

Then *z ∈ Ik,* for every *k ∈* N*,* and therefore *z ∈ ∩Ik∈*N*. ⊓⊔*

**Remark.** Theorem 2.11 is no longer true if all the closed intervals are un- bounded. We can show it considering *Ik* = [*k,∞*[*, k ∈* N*. △*

**Remark.** We note that we used Axiom (R16) to prove Theorems 2.5 and 2.11. It can be shown that (R1) – (R15) together with Theorems 2.5 and 2.11 imply (R1) – (R15) and Theorems 2.1 and 2.2, [45, p. 22]. *△*

**1.2.3 Elements of algebra**

**Theorem 2.12*.*** *For every real x there exists a unique integer k such that k ≤ x<k* + 1*.*

Let *x* be a real number. Its *floor* is the unique (Theorem 2.12) integer *k* satisfying

*k ≤ x<k* + 1*,*

and is denoted *⌊x⌋* (read “the floor of *x*”). Equivalently, the floor of a real number *x* is the largest integer less than or equal to *x.* In some papers the floor of a real number is referred to as the *integer part* of it [33].

The *floor function* is defined by

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R *∋ x ↦→ ⌊x⌋ ∈* Z*.*

Let *x* be a real number. Its *ceiling* is the unique (Theorem 2.12) integer *k* satisfying

*k −* 1 *< x ≤ k,* and is denoted *⌈x⌉* (read “the ceiling of *x*”). Equivalently, the ceiling of a real number *x* is the smallest integer greater than or equal to *x.* The *ceiling function* is defined by

R *∋ x ↦→ ⌈x⌉ ∈* Z*.* Note*⌊√*2*⌋* = 1*, ⌈√*2*⌉* = 2*,*

⌊12⌋

= 0*,*

⌈12⌉

= 1*,*

⌈*−*12⌉

= 0*.*

Obviously, for a real *x,*

*x −* 1 *< ⌊x⌋ ≤ x ≤ ⌈x⌉ < x* + 1*, x ∈* Z *⇐⇒ ⌊x⌋* = *⌈x⌉, x /∈* Z *⇐⇒ ⌈x⌉* = *⌊x⌋* + 1*.*

For an integer *n* one has ⌊*n*2⌋

+

⌈*n*2⌉

= *n,*

and for any nonzero integers *a* and *b*

*⌊⌊n/a⌋/b⌋* = *⌊n/*(*ab*)*⌋* and *⌈⌈n/a⌉/b⌉* = *⌈n/*(*ab*)*⌉.*

The *fractional part* of a real number *x* is defined as *x−⌊x⌋* and is denoted by *{x}.* So, the *fractional part function* is defined by

R *∋ x ↦→ {x} ∈* [0*,*1[*.*

For a real number *x* we denote by *{{x}}* the *distance from x to the nearest integer*. Hence *{{x}}* = min*{{x},⌈x⌉ − x}.*

**Theorem 2.13*.*** *For every two real numbers x and y such that x<y there exists a rational lying between them, that is, x<u<y, for a certain u ∈* Q*.*

*Proof.* Based on Archimedes’ principle (Theorem 2.5) for the positive real *y − x* there exists a natural *n* such that 1 *< n*(*y − x*)*.* Then

1*/n < y − x.* (1.8)

From Theorem 2.12 it follows that there exists an integer *m* such that

*m ≤ nx < m* + 1*.* (1.9)

Obviously, *u* = (*m*+1)*/n* is a rational, and satisfies *x < u.* From the left-hand side of (1.9) as well as from (1.8) we infer that *u* also satisfies

*u* = *mn* + *n* 1*≤ x* + *n* 1*< y. 2*

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An *irrational* number is precisely a nonrational real number; that is, it belongs to R *\* Q*.*

**Corollary 2.6*.*** *Given any two real numbers x and y such that x < y, there exists an irrational number v such that x<v<y. Proof.* Choose any irrational number *v*0 (*√*2*,* for example). Then *x − v*0 *< y−v*0*.* By Theorem 2.13, there exists a rational *u* such that *x−v*0 *<u<y−v*0; that is, *x<v*0 + *u < y.* We remark that *v* = *v*0 + *u* is irrational, because otherwise it follows that *v*0 itself is rational, and this is not the case. *⊓⊔*

A real number is said to be an *algebraic number* if it is a root of a poly- nomial equation with integer coefficients.

A real number is said to be an *transcendental number* if it is not a root of any polynomial equation with integer coefficients.

Let *P*(*x*) be a polynomial of degree *n*

*P*(*x*) = *anxn* + *···* + *a*1*x* + *a*0*,*

with *a*0*,...,an ∈* Z*,* and *an* = 0*,* and *x ∈* R*.*

**Theorem 2.14*.*** *If an irreducible fraction p/q is a root of P, p divides a*0 *and q divides an.*

*Proof.* Suppose *p/q* is a root of *P*; then *qnP*(*p/q*)=0*,* hence

*a*0*qn* + *a*1*qn−*1*p* + *···* + *an−*1*qpn−*1 = *−anpn.*

From this identity *q* divides *anpn,* but *p* and *q* are relatively prime, so *q* divides *an.* Similarly one can show that *p* divides *a*0*. ⊓⊔*

**Corollary 2.7*.*** *The real roots of the polynomial*

*xn* + *an−*1*xn−*1 + *···* + *a*1*x* + *a*0*,*

*are either integers or irrational numbers.*

**Corollary 2.8*.*** *The real roots of the polynomial*

*x*2 *− m* = 0*,*

*where m is prime, are irrational numbers.*

Let *A* and *B* be two sets. If there exists a bijective mapping from *A* onto *B,* we say that *A* and *B* have the same *cardinal number* or that *A* and *B* are *equivalent*, and we write *A* ∼ *B.*

**Theorem 2.15*.*** *The relation* ∼ *defined above is an equivalence relation.*

Recall, for every positive integer *n,* N*∗n* is the set whose elements are precisely the integers 1*,*2*,...,n.* For a set *A* we say that

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(a) *A* The is number *finite* if of *A* elements ∼ N*∗n* In this case we write for some *n* (the empty set is, by definition, finite). of a nonempty *|A|* = *n,* and finite set *A* is *n* provided we read “the number of elements *A* ∼ N*∗n*of *.*

the nonempty and finite set *A* is equal to *n*” or “the cardinality of *A* is *n.*” By definition, *|∅|* = 0*.* (b) *A* is *infinite* if *A* is not finite. (c) *A* is *countable* if *A* ∼ N*∗.* Obviously N∼N*∗.* We write *|A|* = *א*0 and read “the cardinality of *A* is aleph zero.” Aleph is the first letter of the Hebrew alphabet. (d) *A* is *uncountable* if *A* is neither finite nor countable. We write *|A|≥א*1

*> א*0*.* (e) *A* is *at most countable* (or *denumerable*) if *A* ∼ N*∗ n ∈* N*∗.* We write *|A|≤א*0*.*

or *A* ∼ N*∗n* for some

**Remarks.** (a) For two finite sets *A* and *B* so that *B ⊂ A* we have *A* ∼ *B* if and only if *A* = *B.* For infinite sets, however, this is not exactly so. Indeed, let *M* be the set of all even positive integers, *M* = *{*2*,*4*,*6*,...}.* It is clear that *M* is a proper subset of N*∗.* But N*∗* ∼ *M,* because N*∗ ∋ n ↦→* 2*n ∈ M* is a bijection. (b) The sets *{*1*,−*1*,*2*,−*2*,*3*,−*3*,...}* and *{*0*,*1*,−*1*,*2*,−*2*,*3*,−*3*,...}* are equivalent. Indeed, the function

*bk* =

{0*, k* = 1*, ak−*1*, k >* 1*,*

maps the *k*th rank term *ak* of the first set to the (*k* + 1)th rank term in the second set in a bijective way.

From (a) and (b) we conclude that the sets N*∗* and Z are equivalent. Therefore we write *|*Z*|* = *א*0*. △*

**Theorem 2.16*.*** *Every infinite subset of a countable set is countable.*

**Theorem 2.17*.*** *Let {An | n* = 1*,*2*,...} be a countable family of countable sets, and put*

*B* = *∪∞n*=1*An. Then B is countable.*

*Proof.* Let every set *An* be arranged in a sequence (*xn k*)*k, n* = 1*,*2*,...,* and consider the infinite array in which the elements of *An* form the *n*th row, Figure 1.10.

The array contains all elements of *B.* As indicated by the arrows, these elements can be arranged in a sequence

*x*11*, x*21*, x*12*, x*31*, x*22*, x*13*, ... .* (1.10)

If any two of the elements *An* have elements in common, these will appear more than once in (1.10). Hence there is a subset *C* of *B* such that *C* ∼ *B,*

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x11 x12 x13 x14 ... A1

x21 x22 x23 x24 ... A2

x31 x32 x33 x34 ... A2

x41 x42 x43 x44 ... A4

...

**Fig. 1.10.** Infinite array

which shows that *B* is at most countable (Theorem 2.16). Because *A*1 *⊂ B,* and *A*1 is infinite, *B* is infinite, and thus countable. *⊓⊔*

**Corollary 2.9*.*** *Suppose A is at most countable and for every α ∈ A, Bα is at most countable. Put*

*C* = *∪α∈ABα. Then C is at most countable.*

*Proof.* For *C* is equivalent to a subset of *∪αBα. ⊓⊔*

**Theorem 2.18*.*** *Let A be a countable set and let Bn* = *An, for some n ∈* N*∗. Then Bn is countable.*

*Proof.* That *B*1 is countable is obvious, because *B*1 = *A.* Suppose *Bn−*1 is countable (*n* = 2*,*3*,...*)*.* The elements of *Bn* are of the form

(*b, a*) (*b ∈ Bn−*1*, a ∈ A*)*.*

For every fixed *b,* the set of pairs (*b, a*) is equivalent to *A,* and hence, count- able. Thus *Bn* is a countable union of countable sets. By Theorem 2.17, *Bn* is countable. *⊓⊔*

**Corollary 2.10*.*** (Cantor) *The set of all rational numbers is countable.*

*Proof.* We apply Theorem 2.18 with *n* = 2*,* noting that every rational *r* is of the form *a/b,* where *a* and *b* are integers and *b* = 0*.* The set of such pairs (*a, b*)*,* and therefore the set of fractions *a/b,* are countable. *⊓⊔*

Therefore we write *|*Q*|* = *א*0*.*

**Corollary 2.11*.*** *Each set of the form* ]*a, b*[*∩*Q*, where a, b ∈* Q *and a < b, is countable.*

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*Proof.* Equivalently, we prove that between any two different rational numbers there are countably many rational numbers.

The set of positive rational numbers is countable (otherwise, the set of rational numbers is finite). To each positive rational number *q* we assign (uniquely) a pair (*m, n*) of positive integers with no common divisor such that *q* = *m/n.* The map (*m, n*) *→* (*ma* + *nb*)*/*(*m* + *n*) is one-to-one, (*ma* + *nb*)*/*(*m* + *n*) *∈* Q*,* and *a <* (*ma* + *nb*)*/*(*m* + *n*) *< b.* Then to each positive rational number we assigned uniquely a rational number member of the open interval ]*a, b*[*.*

On the other hand, to each rational *p ∈*]*a, b*[ we can find a pair (*m, n*) of positive integers with no common divisors so that *p* = (*ma* + *nb*)*/*(*m* + *n*)*b. m/n* is a positive rational. Thus there is a bijection between the set of positive rational numbers and the set of rational numbers belonging to ]*a, b*[*. ⊓⊔*

**Theorem 2.19*.*** *Let A be the set of all sequences whose elements are the digits* 0 *and* 1*. Then A is uncountable.*

*Proof.* Let *B* be a countable subset of *A,* and let *B* consist of the sequences *s*1*,s*2*,....* We construct a sequence *s* as follows. If the *n*th digit in *sn* is 1 we let the *n*th digit of *s* be 0*,* and vice versa. Then the sequence *s* differs from every member of *B* in at least one place; hence *s /∈ B.* But clearly *s ∈ A,* so that *B* is a proper subset of *A.*

We have shown that every countable subset of *A* is a proper subset of *A.* It follows that *A* is uncountable (for otherwise *A* would be a proper subset of *A,* which is absurd). *⊓⊔*

**Corollary 2.12*.*** *The interval* [0*,*1] *is uncountable.*

*Proof.* Use the binary representation of the real numbers and apply Theorem 2.19. *⊓⊔*

**Corollary 2.13*.*** *Every interval* [*a, b*] (*a<b*) *is uncountable.*

**Corollary 2.14*.*** (Cantor) *The real number set is uncountable.*

Therefore we write *|*R*|* = *א*1*.*

**Theorem 2.20*.*** (Cantor) *The algebraic number set is countable.*

*Proof.* For every polynomial *P*(*x*) = *anxn* + *an−*1*xn−*1 + *···* + *a*1*x* + *a*0 of degree *n* and with integer coefficients, define its *height h*(*P*) as the integer

*n* + *|an|* + *|an−*1*|* + *···* + *|a*1*|* + *|a*0*|.*

Let *Um* be the set of all the roots to all the polynomials *P* satisfying *h*(*P*) *≤ m.* Obviously, for every integer *m,* the set *Um* is finite. Because the set

*∪∞m*=1*Um*

is countable, we conclude that the algebraic number set is countable. *⊓⊔*

26 1 Sets and Numbers **Remark.** Each number of the form *√p,* where *p* is prime, is irrational and algebraic. *△*

**Corollary 2.15*.*** (Cantor) *The transcendental number set is uncountable.*

*Proof.* Suppose the transcendental number set is countable. Then the union of the algebraic number set and the transcendental number set is countable. This union precisely is the real number set. So, we contradict Corollary 2.20. The conclusion follows. *⊓⊔*

**Remark.** From Theorem 2.20 and Corollary 2.15 it follows that “almost all” real numbers are transcendental. *△*

**1.2.4 Elements of topology on** R

A set *A ⊂* R is said to be *open* if for each *x ∈ A* there is a positive *ε* such that ]*x − ε, x* + *ε*[*⊂ A.* Obviously, R is open. We consider by definition the empty set as being open. It follows immediately that

**Theorem 2.21*.*** *Let O denote the family of all open subsets of* R*. Then*

(a) *∅,*R *∈ O.* (b) *The union of any family of open sets is open.* (c) *The intersection of any finite family of open sets is open.*

**Remark.** We may not relax the finiteness assumption in (c). For every *n ∈* N*∗* the interval ] *−* 1*/n,*1*/n*[ is an open set. The set *{*0*}* containing just the origin is not an open set because it does not contain any open interval. Note that *∩n∈*N*∗*] *−* 1*/n,*1*/n*[= *{*0*}. △*

Let *A* be a nonempty subset of R and *x ∈ A.* Then *x* is an *interior* point of *A* if there is an open set *O* with *x ∈ O ⊂ A.* The set of interior points of a set *A ⊂* R is denoted by int*A.* If int*A* = *∅,* we say that *A* has no interior point. This is the case for *A* = *{*1*}* and *A* = *{*1*/n | n* = 1*,*2*,...}.* Always int*A ⊂ A.*

**Theorem 2.22*.*** *Suppose A, B ⊂* R*. Then*

(a) int*A is the union of all open sets contained by A. Thus* int*A is open.* (b) *A is open if and only if A* = int*A.* (c) *The interior of A is the largest open set (in respect to the inclusion of*

*sets) contained in A.* (d) *A ⊂ B implies* int (*A*) *⊂* int (*B*)*.* (e) int(*A ∩ B*) = int (*A*) *∩* int (*B*)*.* (f) int(int*A*) = int *A.* (g) intR = R*.*

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*Proof.* (a) It follows from the definition of int*A.* (b) Suppose that *A* is open. By definition, for every *x ∈ A* there exists a positive *ε* such that ]*x − ε, x* + *ε*[*⊂ A.* This means that *x ∈* int*A.* Thus *A ⊂* int *A.* The other inclusion is trivial. (c) We have to show that if int*A ⊂ B ⊂ A* and *B* is open, then int*A* = *B.* For, suppose int*A* = *B.* Choose an *x ∈ B \* int *A.* Then there is a positive *ε* such that ]*x − ε, x* + *ε*[*⊂ B* and *B ⊂ A.* Then *x ∈* int*A.* Thus we get a contradiction and the claim is proved. (d) It is trivial. (e) Suppose that *x ∈* int (*A*) *∩* int (*B*)*.* Then there exist open sets *O*1*,O*2 such that *x ∈ O*1 *⊂ A* and *x ∈ O*2 *⊂ B.* Thus *x ∈ O*1 *∩ O*2 *⊂ A ∩ B,* hence *x ∈* int (*A ∩ B*)*.*

Suppose now that *x ∈* int (*A ∩ B*)*.* We can find an open set *O* such that *x ∈ O ⊂* (*A ∩ B*)*.* Then *x ∈ O ⊂ A* and *x ∈ O ⊂ B.* Hence *x ∈* int*A* and *x ∈* int *B.* We conclude *x ∈* int (*A*) *∩* int (*B*)*.* (f) int*A* is open by (a). The conclusion follows by (b). (g) We noticed that R is open. The conclusion follows by (b). *⊓⊔*

**Theorem 2.23*.*** *Every open interval is an open set.*

A set *A ⊂* R is said to be *closed* if СR *A* is open. Then we have the following.

**Theorem 2.24*.*** *Let C denote the family of all closed subsets of* R*. Then*

(a) *∅,*R *∈ C.* (b) *The intersection of any family of closed sets is closed.* (c) *The union of any finite family of closed sets is closed.*

*Proof.* (a) We have СR R = *∅* and СR *∅* = R*.* (b) It follows from (b) of Theorem 2.21 and Theorem 1.4. (c) It follows from (c) of Theorem 2.21 and Theorem 1.4. *⊓⊔*

**Remark.** We may not relax the finiteness assumption in (c). For every N*∗* the interval [1*−*1*/n,*1] is a closed set. The set ]0*,*1] is neither open nor closed. We note that *∪n∈*N*∗*[1 *−* 1*/n,*1] = ]0*,*1]*. △*

**Proposition 2.13*.*** *Consider two sets A, B ⊂* R *such that A is open and B is closed. Then A \ B is open and B \ A is closed.*

*Proof.* We write *A \ B* = *A ∩* СR*B* and *B \ A* = *B ∩* СR*A. ⊓⊔*

For *A ⊂* R*,* the *closure* of *A* is the set

cl*A* = *∩{C | A ⊂ C ⊂* R*, C* is closed*}.*

We remark that *A ⊂* cl*A* always.

The interior and the closure of a set are strongly tied.

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**Theorem 2.25*.*** *Consider A ⊂* R*. Then*

СR (cl*A*) = int (СR *A*)*,* СR (int*A*)=cl(СR *A*)*.*

*Proof.* We prove the first identity only. The proof of the second one runs similarly.

Set an arbitrary *x ∈* СR (cl*A*)*.*

*x /∈* cl*A* =*⇒* exists *C* closed, *A ⊂ C,* such that *x /∈ C* =*⇒* exists *O*(= СR *C*) open, *O ⊂* СR *A, x ∈ O* =*⇒ x ∈* int (СR *A*)*.*

Set an arbitrary *x ∈* int (СR *A*)*.* Then

exists *O* open*, O ⊂* СR *A, x ∈ O* =*⇒* exists *C*(= СR *O*) closed*, A ⊂ C, x /∈ C* =*⇒ x /∈* cl*A* =*⇒ x ∈* СR (cl*A*)*. 2*

**Corollary 2.16*.*** *Suppose A ⊂* R*. Then* int*A* = СRcl (СR*A*) *and* cl*A* = СR(int (СR *A*))*.*

**Theorem 2.26*.*** *Suppose A, B ⊂* R*. Then*

(a) cl*A is a closed set.* (b) *A is closed if and only if A* = cl*A.* (c) cl*A is the smallest closed set (in respect to the inclusion of sets) contain-*

*ing A.* (d) *A ⊂ B implies* cl (*A*) *⊂* cl (*B*)*.* (e) cl(*A ∪ B*)=cl(*A*) *∪* cl (*B*)*.* (f) cl(cl*A*) = cl*A.* (g) clR = R*.*

*Proof.* (a) It follows from the definition of the closure and (b) of Theorem 2.24. (b) If *A* is closed, it appears as a member in the intersection defining the cl*A.* Then cl*A ⊂ A.* Thus *A* = cl*A.*

If *A* = cl*A, A* is closed by (a). (c) We have to show that if *A ⊂ B ⊂* cl*A* and *B* is closed, then *B* = cl*A.* Because *B* is closed and *A ⊂ B, B* appears as a member in the intersection defining the cl*A.* So, *B ⊃* cl*A.* Then *B* = cl*A,* indeed. (d) It is trivial. (e) We apply Theorems 1.4 and 2.25. Then

СRcl (*A ∪ B*) = intСR(*A ∪ B*) = int (СR*A ∩* СR*B*) = int (СR*A*) *∩* int (СR*B*)=(СRcl*A*) *∩* (СRcl*B*) = СR(cl*A ∪* cl*B*)*.*

(f) By (a) cl*A* is a closed set. The conclusion follows by (b). (g) The closure of any set is contained in R*.* So, R *⊂* clR *⊂* R*. ⊓⊔*

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**Proposition 2.14*.*** *Consider two open and disjoint sets A, B ⊂* R*. Then*

(a) *The closure of one set does not intersect the other; that is, B ∩* cl*A* =

cl (*B*) *∩ A* = *∅.* (b) (intcl *A*) *∩* (int cl *B*) = *∅.*

*Proof.* (a) From the hypothesis we have *A ∩ B* = *∅.* Then *B ⊂* СR *A* and cl*B ⊂* СR *A.* Finally, cl*B ∩ A* = *∅.* (b) From (a) we have that *A ∩* cl*B* = *∅.* Then *A ∩* (int cl *B*) = *∅.* Because int cl *B* is open, we repeat the reasoning to this last open set and to *A. ⊓⊔*

A *neighborhood* of a point *x ∈* R is any set *A ⊂* R containing an open interval *O* with *x ∈ O* (i.e., *x ∈ O ⊂ A*)*.* The system of all neighborhoods of a point *x ∈* R is denoted by *V*(*x*)*.* Obviously, R *∈ V*(*x*)*,* for each *x ∈* R*.*

**Proposition 2.15*.*** *For a set A ⊂* R *the following two sentences are equiva- lent.*

(a) *x ∈* cl*A.* (b) *For every V ∈ V*(*x*) *one has V ∩ A* = *∅.*

*Proof.* We show that (a) implies (b). Suppose that for a point *x* and an open neighborhood *V ∈ V*(*x*)*, V ∩ A* = *∅.* It means that *A ⊂* СR*V* and СR*V ∈ C.* Then cl*A ⊂* СR*V* and *x /∈* cl*A,* thus (a) does not hold. We show that (b) implies (a). Suppose that *x /∈* cl*A.* Then there exists a closed set *C* with *A ⊂ C* and *x /∈ C.* Set *V* = СR*C* and remark that *V* is open, *x ∈ V,* and *V ∩ A* = *∅.* But this contradicts (b). *⊓⊔*

**Corollary 2.17*.*** *Suppose V is an open set and V ∩ A* = *∅. Then V ∩* cl*A* = *∅.*

*Proof.* Suppose there is an element *x ∈ V ∩*cl*A.* Then by Proposition 2.15 it follows that *V ∩ A* = *∅. ⊓⊔*

For *x ∈* R and *A ⊂* R we say that *x* is a *limit point* or *accumulation point* of *A* if every *V ∈ V*(*x*) contains some point of *A* distinct from *x*; that is, *V ∩* (*A \ {x}*) = *∅.* Denote by *A′* the set of limit (accumulation) points of the set *A.*

**Theorem 2.27*.*** *Consider A,B,Aα ⊂* R*, α ∈ I. Then* (a) cl*A* = *A ∪ A′.* (b) *A ⊂ B implies A′ ⊂ B′.* (c) (*A ∪ B*)*′* = *A′ ∪ B′.* (d) *∪α∈IA′α* = (*∪α∈IAα*)*′.* **Corollary 2.18*.*** *Consider A ⊂* R*. Then A is closed if and only if A′ ⊂ A.*

**Theorem 2.28*.*** *Let A be a closed set of real numbers that is bounded above. Set y* = sup*A. Then y ∈ A.*

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*Proof.* Suppose *y /∈ A.* For any *ε >* 0 there is a point *x ∈ A* such that *y−ε<x<y,* Theorem 2.7, page 18. Thus, every neighborhood of *y* contains a point *x ∈ A,* and *x* = *y,* because *y /∈ A.* It follows that *y* is a limit point of *A* which is not a point of *A,* so that *A* is not closed. This contradicts the hypothesis. *⊓⊔*

**Corollary 2.19*.*** *Let B be a closed set of real numbers that is bounded below. Set y* = inf *B. Then y ∈ B.*

*Proof.* Set *A* = *−B* = *{−b | b ∈ B}.* Then *A* is closed and bounded above. Moreover, *y* = *−*sup*A.* Then *y ∈ B. ⊓⊔*

A point *x ∈* R is said to be a *closure point* or (*adherent point*) of the set *A ⊂* R if every neighborhood of *x* has a nonempty intersection with *A.*

**Theorem 2.29*.*** *Suppose ∅ ̸*= *A ⊂* R*. Then A is closed if and only if A coincides with the set of its closure points.*

*Proof.* Denote by *B* the set of closure points of *A.*

Suppose *A* = *B.* Then СR*A* = СR*B* and for each *x ∈* СR*A* we can find a neighborhood *V* of *x* such that *V ∩ A* = *∅*; that is, *V ⊂* СR*A.* Thus *x* is an interior point of СR*A.* The point *x* has been chosen arbitrary, so the set СR*A* is open. Hence *A* is closed.

We have that *A ⊂ B.* Suppose that *A* is closed. Then СR*A* is open. For any *x /∈ A,* СR*A* is a neighborhood of *x* not intersecting *A.* Then *x /∈ B.* Thus we get that *B ⊂ A,* and finally *A* = *B. ⊓⊔*

The *frontier* or *boundary* of a set *A ⊂* R is the set (cl*A*) *∩* (clСR*A*)*.* We denote it by fr *A* and we note that it is closed. We remark that fr [0*,*1] = fr ]0*,*1[ = fr*{*0*,*1*}* = *{*0*,*1*}.*

**Theorem 2.30*.*** *Suppose A ⊂* R*. Then A is open if and only if A ∩* fr *A* = *∅.*

**Theorem 2.31*.*** *Suppose A, B ⊂* R*. Then we have*

(a) int*A* = *A \* fr*A.* (e) fr (R *\ A*) = fr*A.* (b) cl*A* = *A ∪* fr*A.* (f) *A ∩ A* = *A.* (c) fr (*A ∪ B*) *⊂* fr*A ∪* fr*B.* (g) R = int*A ∪* fr*A ∪* int (R *\ A*)*.* (d) fr (*A ∩ B*) *⊂* fr*A ∪* fr*B.* (h) fr (int*A*) *⊂* fr*A.*

(i) *A is open if and only if* fr*A* = cl*A \ A.* (j) *A is closed if and only if* fr*A* = *A \* int*A.*

*Proof.* (a) Successively we have

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*A \* fr*A* = *A \* ((cl*A*) *∩* (clСR*A*)) = (*A \* cl*A*) *∪* (*A \* clСR*A*)

= *A \* clСR*A* = *A ∩* int*A* = int*A.* (c) We have

fr (*A ∪ B*) = (cl (*A ∪ B*)) *∩* clСR(*A ∪ B*) = (cl*A ∪* cl*B*) *∩* cl (СR*A ∩* СR*B*) *⊂* (cl*A∪*cl*B*) *∩* clСR*A ∩* clСR*B* = (cl*A ∩* (clСR*A*)) *∪* (cl*B ∩* (clСR*B*))

= fr*A ∪* fr*B.*

A subset *A* of R is said to be *dense* in R provided cl*A* = R*.* Invoking Theorem 2.13, we find the following.

**Corollary 2.20*.*** Q *is dense in* R*.*

**1.2.5 The extended real number system**

The *extended real number set* consists of the real number set to which two symbols, +*∞*(= *∞*) and *−∞* have been adjoined, with the following prop- erties. (a) If *x* is real, *−∞ <x<* +*∞,* and

*x* + *∞* = *∞* + *x* = +*∞, x − ∞* = *−∞* + *x* = *−∞, x*+*∞* = *x−∞* = 0*.* (b) If *x >* 0*, x*(+*∞*)=(*∞*)*x* = +*∞, x*(*−∞*)=(*−∞*)*x* = *−∞.* (c) If *x <* 0*, x*(+*∞*)=(*∞*)*x* = *−∞, x*(*−∞*)=(*−∞*)*x* = +*∞.* The extended real number system is denoted by R = R*∪{*+*∞} ∪ {−∞}*with the above-mentioned conventions.

Any element of R is called *finite* whereas +*∞* and *−∞* are called *infini- ties*.Let *A* be a nonempty subset of the extended real number set. If *A* is not bounded above (i.e., for every real *y* there is an *x ∈ A* such that *y<x*), we define sup*A* = +*∞.* Similarly, if *A* is not bounded below (i.e., for every real *y* there is an *x ∈ A* such that *y>x*), we define inf *A* = *−∞.*

By definition we takesup*∅* = *−∞* and inf *∅* = *∞.*

We define intervals involving infinities

[*a,*+*∞*[ = *{x ∈* R *| a ≤ x}.* ]*a,*+*∞*[ = *{x ∈* R *| a<x}.* [*a,*+*∞*] = *{x ∈* R *| a ≤ x ≤* +*∞}.* ]*a,*+*∞*] = *{x ∈* R *| a<x ≤* +*∞}.* ] *− ∞,a*] = *{x ∈* R *| x ≤ a}.* ] *− ∞,a*[ = *{x ∈* R *| x<a}.*

[*−∞,a*] = *{x ∈* R *| −∞ ≤ x ≤ a}.* [*−∞,a*[ = *{x ∈* R *| −∞ ≤ x<a}.* [*−∞,∞*] = R*.* ] *− ∞,∞*[ = R*.*

By definition, an open neighborhood of +*∞* is an open interval ]*a,*+*∞*[*,* for some *a ∈* R*.* Similarly, an open neighborhood of *−∞* is an open interval ] *− ∞,a*[*,* for some *a ∈* R*.*

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**Theorem 2.32*.*** *Every subset of* R *has a unique supremum and has a unique infimum, these belong to* R*.*

We define the *Minkowski*8 *operations* on subsets of R*.* For *A, B* nonempty subsets of R*,* define *A* + *B* = *{a* + *b | a ∈ A* and *b ∈ B}.* It one of the two sets is empty, their Minkowski sum is empty. For *A* a nonempty subset of R*,* and *p* a real number, define *p* + *A* = *{p* + *a | a ∈ A}* and *pA* = *{pa | a ∈ A}.* If *A* = *∅,* then *p* + *A* = *pA* = *∅.*

**Theorem 2.33*.*** *Let A, B be subsets of* R*, let a by a real number, and let c >* 0*. Then*

*A ⊂ B* =*⇒* sup*A ≤* sup*B,* inf *A ≥* inf *B.* (1.11) sup(*−A*) = *−*inf *A,* inf(*−A*) = *−*sup*A.* (1.12) sup(*A* + *a*) = sup*A* + *a,* inf(*A* + *a*) = inf *A* + *a.* (1.13) sup(*cA*) = *c*sup*A,* inf(*cA*) = *c*inf *A.* (1.14) sup(*A* + *B*) = sup*A* + sup*B,* inf(*A* + *B*) = inf *A* + inf *B,* (1.15)

whenever the operations are well *−* defined*.*

*Proof.* Suppose that *B* = *∅.* Then sup*A* = sup*B* = *−∞* and inf *A* = inf *B* = *∞.*

Suppose that *∅* = *A ⊂ B* = *∅.* Then *−∞* = sup*A <* sup*B* and *∞* = inf *A >* inf *B.*

Suppose that *∅ ̸*= *A ⊂ B.* If *A* = *B,* the conclusions are obvious. Therefore we take into account that *A* = *B.* For each *a ∈ A* we can find an element *b ∈ B* such that *a ≤ b.* Moreover, *a ≤ b ≤* sup*B.* Hence sup*B* is an upper bound for *A* and thus sup*A ≤* sup*B.* Similarly, we can show the second conclusion in (1.11).

If *A* = *∅,* the equalities in (1.12) are true. Suppose that *A* = *∅* and that *A* is unbounded below. Then *−A* is unbounded above. Thus *∞* = sup(*−A*) = *−*inf *A.* We now suppose that *A* is bounded below, denote *α* = inf *A,* and apply the characterizations of supremum and of infimum given by Theorem 2.7, respectively, Theorem 2.9. Then *α ≤ a* for all *a ∈ A* and for every *ε >* 0 there exists *aε ∈ A* such that *aε < a* + *ε.* Immediately follows that *−α ≥ −a* for all *−a ∈ −A* and for every *ε >* 0 there exists *−aε ∈ −A* such that *−aε > −a − ε.* Thus *−α* = sup(*−A*)*.* Similarly, we can show the second conclusion in (1.12).

If *A* = *∅,* the equalities in (1.13) are true. Suppose that *A* = *∅, a* = 0*,* and that *A* is unbounded above. Then *A* + *a* is also unbounded above and the both sides of the first equality in (1.13) are equal to *∞.* If *A* is bounded above, then *A* + *a* is also bounded above. Using the characterization of the supremum offered by Theorem 2.7, we can check that the first equality in (1.13) is true. Similarly, we can show the second conclusion in (1.13).

8 Hermann Minkowski, 1864–1909.

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If *A* = *∅,* the equalities in (1.14) are true. Suppose that *A* = *∅* and that *A* is unbounded above. Then *cA* is also unbounded above and the both sides of the first equality in (1.14) are equal to *∞.* If *A* is bounded above, then *cA* is also bounded above. By the characterization in Theorem 2.7, we can check that the first equality in (1.14) is true. Similarly, we can show the second conclusion in (1.14).

If *A* = *∅,* so is *A* + *B,* and the first equality in (1.15) reduces to *−∞* + sup*B* = *−∞.* This is true since the case *−∞* + *∞* is excluded. The case *A* = *B* = *∅* gives us an equality. We now suppose that both sets *A* and *B* are nonempty. Then *supA* = *α* and sup*B* = *β,* for some *α, β ∈*] *− ∞,∞*]*.* Then for every *x ∈ A* and *y ∈ B, x*+*y ≤ α*+*β.* This means that sup(*A*+*B*) *≤ α*+*β,* this is sup(*A*+*B*) *≤* sup*A*+sup*B.* Hence a half of the first equality in (1.15) is proved. At the same time if sup(*A* + *B*) = *∞,* then the equality holds. Suppose that sup(*A* + *B*) *< ∞* and denote *γ* = sup(*A* + *B*)*.* Then for all *x ∈ A* and *y ∈ B, x* + *y ≤ γ,* or *x ≤ γ − y.* Immediately follow that sup*A ≤ γ − y* and *y ≤ γ −* sup*A.* Therefore sup*B ≤ γ −* sup*A* and we conclude that sup*A* + sup*B ≤* sup(*A* + *B*)*.* Now the first equality in (1.15) proved. Similarly, we can prove the second equality in (1.15). The second equality in (1.15) can be proved in the following way

inf(*A* + *B*) = *−*sup(*−*(*A* + *B*)) = *−* sup(*−A − B*) = *−*sup(*−A*) *−* sup(*−B*)

= inf *A* + inf *B,*

whenever the operations are well-defined. *⊓⊔*

**1.2.6 The complex number system**

Traditionally, there are two ways to introduce the set of *complex numbers,* denoted as C*.* One way consists in defining a complex number as a pair of real numbers and afterwards two operations are defined, an addition and a multiplication. Thus we get that C together with the two operations is a field. This way is followed in [127].

More traditionally and more intuitively is considering a complex number as an algebraic entity of the form *a*+*ib,* where *i*2 = *−*1 and *a, b ∈* R*.* Denote *z* = *a* + *ib.* Then *a* is its *real part* denoted Re *z,* and *b* is its *imaginary part*, denoted Im*z.* The *absolute value* of *z* is *|z|* = *√a*2 + *b*2*.*

A complex number *z* = *a* + *ib* is null if and only if *a* = *b* = 0*.* A nonnull complex number *z* = *a* + *ib* can be written as

*z − a* + *ib* = *|z|*( *a|z|* + *i b|z|*)

= *|z|*(cos *θ* + *i*sin*θ*)

for a certain *θ.* Then the *Moivre* 9 *formula*

9 Abraham de Moivre, 1667–1754.

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*z* = cos*θ* + *i* sin*θ* =*⇒ zn* = cos *nθ* + *i*sin*nθ*

is valid.

The properties of complex numbers are introduced in many textbooks. We suggest one of them [127].

**1.3 Exercises**

**1.1.** For arbitrary sets *A, B,* and *C,* show that

(i) (*A \ B*) *∩ B* = *∅.* (ii) *A \* (*A \ B*) = *A ∩ B.* (iii) (*A \ B*) *∪* (*B \ A*)=(*A ∪ B*) *\* (*A ∩ B*)*.* (iv) (*A \ B*) *\ C* = *A \* (*B ∪ C*)*.*

(v) (*A \ B*) *∩ C* = (*A ∩ C*) *∩* (*C \ B*)=(*A ∩ C*) *\* (*B ∩ C*)*.*

**1.2.** Show that the inclusion *A \ B ⊂ C* holds if and only if *A ⊂ B ∪ C.*

**1.3.** For arbitrary sets *A, B,* and *C,* show that

(i) (*A ∪ B*) *\ C* = (*A \ C*) *∪* (*B \ C*)*.* (ii) (*A ∩ B*) *\ C* = (*A \ C*) *∩* (*B \ C*)*.* (iii) (*A ∪ C*) *\ B ⊂* (*A \ B*) *∪ C.*

**1.4.** Find the mutual relationships between sets *X* and *Y* (i.e., *X ⊂ Y, X* = *Y,* or *X ⊃ Y* ) provided

(i) *X* = *A ∪* (*B \ C*)*, Y* = (*A ∪ B*) *\* (*A ∪ C*)*.* (ii) *X* = (*A ∩ B*) *\ C, Y* = (*A \ C*) *∩* (*B \ C*)*.* (iii) *X* = *A \* (*B ∪ C*)*, Y* = (*A \ B*) *∪* (*A \ C*)*.* (iv) *X* = (*A × B*) *∪* (*C × B*)*, Y* = (*A ∪ C*) *× B.*

(v) *X* = (*A × B*) *∪* (*C × D*)*, Y* = (*A × C*) *∪* (*B × D*)*.* (vi) *X* = (*A ∩ B*) *×* (*C ∩ B*)*, Y* = (*A × C*) *∩* (*B × D*)*.* (vii) *X* = (*A ∪ B*) *×* (*C ∪ B*)*, Y* = (*A × C*) *∪* (*B × D*)*.*

**1.5.** Let *An, n ∈* N*,* be a collection of sets.

(i) Consider *Bn* = *∪ni*=0*Ai, n ∈* N*.* Show that *∪∞n*=0*Bn* = *∪∞n*=0*An.* (ii) Consider *Bn* = *∩ni*=0*Ai, n ∈* N*.* Show that *∩∞n*=0*Bn* = *∩∞n*=0*An.*

**1.6.** Let *Am,n, m,n ∈* N*,* be a collection of sets. Which inclusion does it hold

(i) *∪∞m*=0(*∩∞n*=0*Am,n*) *⊂ ∩∞n*=0(*∪∞m*=0*Am,n*); (ii) *∪∞m*=0(*∩∞n*=0*Am,n*) *⊃ ∩∞n*=0(*∪∞m*=0*Am,n*)?

**1.7.** For any two sets *A* and *B* show that

(i) С*X∅* = *X* and С*XX* = *∅.* (ii) С(*A \ B*)=(С*A*) *∪ B.*

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(iii) (*A ∩* С*B*) *∪* (С*A ∩ B*)=(*A ∪ B*) *∩* (С*A ∪* С*B*)=(*A ∪ B*) *\* (*A ∩ B*)*.* (iv) *A ⊂ B* =*⇒* С*B ⊂* С*A.*

**1.8.** There are given three sets *A, B,* and *U* with *A ⊂ U* and *B ⊂ U.* Find the set *X ⊂ U* satisfyingС(*X ∪ A*) *∪* (*X ∪* С*A*) = *B.*

**1.9.** Show that

(i) *P*(*A*) *∪ P*(*B*) *⊂ P*(*A ∪ B*)*.* (ii) *P*(*A*) *∪ P*(*B*) = *P*(*A ∪ B*) *⇐⇒ A ⊂ B* or *B ⊂ A.* (iii) *P*(*A*) *∩ P*(*B*) = *P*(*A ∩ B*)*.* (iv) *P*(*A \ B*) *⊂ P*(*A*) *\ P*(*B*)*.*

**1.10.** Show whether each of the following functions is one-to-one and/or onto.

(i) Function *f* : N *→* N*,* defined by *f*(*n*)=2*n, n ∈* N*.* (ii) Function *f* : Q *×* Q *→* Q*,* defined by *f*(*p, q*) = *p, p,q ∈* Q*.* (iii) Function *f* : Q *×* Q *→* Q *×* Q*,* defined by *f*(*p, q*)=(*p,−q*)*, p,q ∈* Q*.*

**1.11.** Let *f* : *A → B* and *g* : *B → C* be functions. Then the following statements are true.

(i) If *f* and *g* are one-to-one, then *g ◦ f* is one-to-one. (ii) If *f* and *g* are onto, then *g ◦ f* is onto. (iii) If *g ◦ f* is one-to-one, then *f* is one-to-one. (iv) If *g ◦ f* is onto, then *f* is onto.

**1.12.** Let *A* and *B* be sets and *f* : *A → B* be a mapping. Consider two mappings *f∗* : *P*(*A*) *→ P*(*B*) and *f ∗* : *P*(*B*) *→ P*(*A*) defined by *f∗*(*M*) = *f*(*M*)*,* respectively, *f∗*(*N*) = *f −*1(*N*)*,* where *M ⊂ A* and *N ⊂ B.*

(a) Show that the following sentences are equivalent.

(i) *f* is one-to-one. (ii) *f∗* is one-to-one. (iii) *f ∗* is onto. (iv) *f*(*M ∩ N*) = *f*(*M*) *∩ f*(*N*)*,* for every *M,N ∈ P*(*A*)*.* (v) *f*(С*AM*) *⊂* С*Bf*(*M*)*,* for every *M ∈ P*(*A*)*.* (b) Show that the following sentences are equivalent.

(i) *f* is onto. (ii) *f∗* is onto. (iii) *f ∗* is one-to-one. (iv) С*Bf*(*M*) *⊂ f*(С*AM*)*,* for every *M ∈ P*(*A*)*.* (c) Show that the following sentences are equivalent.

(i) *f* is bijective. (ii) *f∗* is bijective. (iii) *f ∗* is bijective. (iv) *f*(С*AM*) = С*Bf*(*M*)*,* for every *M ∈ P*(*A*)*.*

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**1.13.** Let *A* and *B* be sets and *f* : *A → B* a mapping.

(a) Show that the following sentences are equivalent.

(i) *f* is one-to-one. (ii) For any two functions *g,h* : *C → A* with *f ◦ g* = *f ◦ g,* it follows that *g* = *h.* (b) Show that the following sentences are equivalent.

(i) *f* is onto. (ii) For any two functions *g,h* : *B → C* with *g ◦ f* = *h ◦ f,* it follows that *g* = *h.*

**1.14.** Let *a* and *b* nonnegative numbers. Show that *a ≤ b* if and only if *aa ≤ bb.*

**1.15.** Let *a* and *b* nonnegative numbers and *n ∈* N*∗.* Show that *a ≤ b* if and only if *an ≤ bn.*

**1.16.** Show that the following sets are countable.

(i) *{*2*n | n ∈* N*∗}.* (ii) The set of triangles in plane whose vertices have rational coordinates. (iii) The set of points in the plane having rational coordinates. (iv) The set of polynomials having rational coefficients.

**1.17.** Consider a set *M* having *m* elements and a set *N* having *n* elements, *m, n ≥* 1*.* Show that

(i) The number of functions *f* : *M → N* is equal to *nm.* (ii) If *m* = *n,* the number of bijective functions from *M* to *N* is equal to *m*!*.* (iii) If *m ≤ n,* the number of one-to-one functions from *M* to *N* is equal to

*n*(*n −* 1)(*n −* 2)*···*(*n − m* + 1)*.* (iv) If *m ≥ n,* the number of onto functions from *M* to *N* is equal to

*nm −*

(*n*1)(*n −* 1)*m* +

(*n*2)(*n−* 2)*m −*

(*n*3)(*n −* 3)*m* + *···* +(*−*1)*n−*1( *n*

*n −* )1*.*

**1.18.** Prove the following identities

(*n*0)

+

(*n*1)

+

(*n*2)

+ *···* +

( *n − n*

1)

+ ((*nnn*)

= 2*n, n ∈* N*∗,* 0)

+

(*n*2)

+

(*n*4)

+ *···* = 2*n−*1*, n ∈* N*∗,* (*n*1)

+

(*n*3)

+

(*n*5)

+ *···* = 2*n−*1*, n ∈* N*∗.*

1.3 Exercises 37

**1.19.** Consider a set *S* containing *n* elements. The *n*th Bell 10 number *Bn* is defined as the number of partitions of *S.* If *S* = *∅,* by definition *B*0 = 1*.* Show that

*B*1 = 1*, B*2 = 2*, B*3 = 5*, B*4 = 15*.* (1.16)

*Bn*+1 =1+

∑*nk*=1(*nk*)*Bk* =

∑*nk*=0(*nk*)*Bk.* (1.17)

**1.20.** Suppose *A* is a finite set so that *|A|* = *m.* Show that the number of solutions to

*A* = *A*1 *∪ A*2 *∪···∪ Ak* is equal to (2*k −* 1)*m.*

**1.21.** Consider *A*1*,A*2 *...,An* finite sets. Show that

*|A*1 *∪ A*2 *∪···∪ An|* = ∑*|Ai| −* ∑*i<j*

*|Ai ∩ Aj|* + ∑

*i<j<k*

*|Ai ∩ Aj ∩ Ak| − ...*

*|A*1 *∩ A*2 *∩···∩ An|* = ∑+ *|A*(*−*1)*i| − n*+1∑*|A*1 *∩ A*2 *∩···∩ An|,*

*i<j*

*|Ai ∪ Aj|* + ∑

*i<j<k*

*|Ai ∪ Aj ∪ Ak| − ...*

+ (*−*1)*n*+1*|A*1 *∪ A*2 *∪···∪ An|.*

**1.22.** Let *x, y* be real numbers with *y >* 0*.* Show that

∑

0*≤ k< y*

⌊*x* + *ky*⌋

= *⌊xy* + *⌊x* + 1*⌋* (*⌈y⌉ − y*)*⌋ .*

**1.23.** Let *n ∈* N*.* Then show that

⌊*n* + 1

2

⌋

+

⌊*n* + 2

22

⌋

+ *···* +

⌊*n* + 2*k*

2*k*+1

⌋

+ *···* = *n.* (1.18)

**1.24.** Let *n* be a natural number greater than 1*.* Show that there is no system of three integers of the form *x* = *a − r, y* = *a,* and *z* = *a* + *r,* with *r >* 0*,* such that

*xn* + *yn* = *zn.*

**1.25.** Let *A* be a finite set. Then show that *|P*(*A*)*|* = 2*|A|.*

**1.26.** (Square root inequality) For *x ≥* 1*,* show that

2*√x* + 1 *−* 2*√x < √*1*x <* 2*√x −* 2*√x −* 1*.* (1.19)

10 Eric Temple Bell, 1883–1960.

38 1 Sets and Numbers

**1.27.** (Schur 11 inequality) Let *x, y,* and *z* be nonnegative numbers. Show that for any *r >* 0*,* we have

*xr*(*x − y*)(*x − z*) + *yr*(*y − z*)(*y − x*) + *zr*(*z − x*)(*z − y*) *≥* 0*.*

**1.28.** (Bernoulli 12 inequality) Consider *n* real numbers *xi ≥ −*1*, i* = 1*,*2*,..., n,* such that all of them have the same sign. Show that

(1 + *x*1)(1 + *x*2)*...*(1 + *xn*) *≥* 1 + *x*1 + *x*2 + *···* + *xn.* (1.20)

**1.29.** (Bernoulli inequality) For every *n ∈* N*∗* and every *x ≥ −*1*,* show that

(1 + *x*)*n ≥* 1 + *nx.* (1.21)

**1.30.** (i) For *n ≥* N*∗,* show that

2 *≤*

(1 + *n*1)*n*

*<* 3*.*

(ii) Show that *n*+1*√n* + 1 *≤ √nn, ∀n ∈* N*∗, n ≥* 3*.*

**1.31.** (Mean inequality) Let *x*1*,x*2*,...,xm* be positive reals. Show that the *geometric mean* is less than or equal to the *arithmetic mean;* that is,

*m√x*1*x*2 *...xm ≤ x*1 + *x*2 + *m ···* + *xm*

*.* (1.22)

**1.32.** Let *x*1*,x*2*,...,xm* be positive reals. Show that the *harmonic mean* is less than or equal to the geometric mean; that is,

*m* 1*x*1 + *x*12 + *···* + *x*1*m*

*≤ m√x*1*x*2 *...xm.* (1.23)

**1.33.** (i) Show that in (1.22) and (1.23) we have equality if and only if *x*1 =

*···* = *xm.* (ii) Let *x*1*,x*2*,...,xn* be positive numbers. Show that

(*x*1 + *x*2 + *···* + *xn*)( 1*x*1 + *x*12 + *···* + *x*1*n*)

*≥ n*2*.*

**1.34.** (Kantorovich 13inequality) Suppose *x*1*,x*2*,...,xn ∈* [*a, b*]*,* 0 *< a ≤ b,* and *ti ≥* 0*.* Show that ( ∑*ni*=1

*tixi*)( ∑*ni*=1

*tixi*)

*≤* (*a* 4*ab*

+ *b*)2

( ∑*i*=1

*nti*)2

*.*

11 Issai Schur, 1875–1941. 12 Johann (I) Bernoulli, 1667–1748. 13 Leonid Vitalyevich Kantorovich (Леонид Витальевич Канторович), 1912–1986.